UNIT- 4
DIFFERENTIAL CALCULUS

Learning Objectives

- To learn concept of function, limit, and differentiation for function of one variables.
- To learn the differentiation of various functions, their sum and product.
- To learn applications of derivatives in real life applications

4.1 FUNCTIONS

<u>Definition of Function</u>: Let A and B be two non empty sets. A rule $f:A \to B$ (read as f from A to B) is said to be a function if to each element x of A there exists a unique element y of B such that $f(x) = y$.y is called the image of x under the map f. Here x is independent variable and y is dependent variable.

There are mainly two types of functions: Explicit functions and Implicit functions. If y is clearly expressed in the terms of x directly then the function is called Explicit function. e.g. $y = x + 20$.

If y can‟t be expressed in the terms of x directly then the function is called Implicit function. e.g. $a x^2 + 2hxy + by^2 = 1$

Functions Types	Algebraic	Trigonometric	Inverse Trigonometric	Exponential	Logarithmic
Examples	$y = x^2 + x + 1$, $y = x^3 - 3x + 2$, $x + 1$, y= etc.	$y = sinx$, $y = cosx$, $y = secx$, etc.	$y = tan^{-1}x$, $y = cos^{-1}x$, $y = cot^{-1}x$, etc.	$y = e^x$, $y = 2^x$, $y = 5^x$, etc.	$y = log_e x$, $y = log_5 x$, $y = log_{10} x$, etc.

We may further categorize the functions according to their nature as:

Even Function: A function $f(x)$ is said to be an even function if $f(-x) = f(x)$ for all x.

For Example: x^2, $x^4 + 1$, $\cos(x)$ etc.

Odd Function: A function $f(x)$ is said to be an odd function if $f(-x) = -f(x)$ for all x.

For Example: x^3, $\sin(x)$, $\tan(x)$ etc.

Periodic Function: A function is said to be a periodic function if it retains same value after a certain period.

For Example: , etc.

As

Therefore is a periodic function with period .

Examples to solve functions:

Example 1. If $f(x) = x^2 + 1$, find $f(2)$.

Sol. Given that $f(x) = x^2 + 1$

Put $x = 2$ in function, we get

$f(2) = 2^2 + 1 = 4 + 1 = 5$.

Example 2. If $f(x) = 2x^2 - 4x + 6$, find $\frac{f(-2)}{f(1)}$.

Sol. Given that $f(x) = 2x^2 - 4x + 6$

Put $x = -2$ in function, we get

$f(-2) = 2(-2)^2 - 4(-2) + 6 = 2(4) + 8 + 6 = 8 + 14 = 22$

Again put $x = 1$ in function, we get

$f(1) = 2(1)^2 - 4(1) + 6 = 2 - 4 + 6 = 4$

Therefore $\frac{f(-2)}{f(1)} = \frac{22}{4} = 5.5$

Note:

(i) The symbol "∞" is called infinity.

(ii) $\frac{a}{0}$ is not finite $(where\ a \neq 0)$ and it is represented by ∞ .

(iii) $\frac{a}{\infty} = 0$ if $(a \neq \infty)$.

Indeterminate Forms: The following forms are called indeterminate forms:

(These forms are meaningless)

Concept of Limits:

A function $f(x)$ is said to have limit l when x tends to a, if for every positive ε (however small) there exists a positive number δ such that $|f(x) - l| < \varepsilon$ for all values of x for which $0 < |x - a| < \delta$ and it is represented as

Some basic properties on Limits:

(i) $\lim_{x \to a} K = K$ where K is some constant.

(ii) $\lim_{x \to a} K \,.\, f(x) = K \,.\, \lim_{x \to a} f(x)$ where is some constant.

(iii) $\lim_{x \to a} [\, f(x) + g(x) \,] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

(iv) $\lim_{x \to a} [\, f(x) - g(x) \,] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

(v) $\lim_{x \to a} [\, f(x) . g(x) \,] = \lim_{x \to a} f(x) . \lim_{x \to a} g(x)$

(vi) $\lim_{x \to a} \left[\frac{f(x)}{g(x)} \right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ provided that $\lim_{x \to a} g(x) \neq 0$

(vii) $\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x) \right]^n$

Methods of finding the limits of the functions:

1) Direct Substitution Method
2) Factorization Method
3) Rationalization Method etc.

Some Standard Limits Formulas:

1) $\lim_{x \to a} \frac{x^n - a^n}{x - a} = n a^{n-1}$

2) $\lim_{x \to \infty} \left(1 + \frac{1}{x} \right)^x = e$

3) $\lim_{x \to 0} (1 + x)^{\frac{1}{x}} = e$

4) $\lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a$

5) $\lim_{x \to 0} \frac{e^x - 1}{x} = \log_e e = 1$

6) $\lim_{x \to 0} \sin x = 0$

7) $\lim_{x \to 0} \tan x = 0$

8) $\lim_{x \to 0} \cos x = 1$

9) $\lim_{x \to 0} \frac{\sin x}{x} = 1$

10) $\lim_{x \to 0} \frac{\tan x}{x} = 1$

Some Solved Examples on limits:

Example 3. Evaluate $\lim_{x \to -1} \left(1 + x + x^2 + x^3\right)$.

Sol. $\lim_{x \to -1} \left(1 + x + x^2 + x^3\right) = 1 + (-1) + (-1)^2 + (-1)^3 = 1 - 1 + 1 - 1 = 0$

Example 4. Evaluate $\lim_{x \to -1} \frac{x^3 + 6}{x + 1}$.

Sol. $\lim_{x \to -1} \frac{x^3 + 6}{x + 1} = \frac{(-1)^3 + 6}{-1 + 1} = \frac{-1 + 6}{0} = \frac{5}{0} = \infty$

Example 5. Evaluate $\lim_{x \to 2} \frac{x^3 - 8}{x - 2}$.

Sol. $\lim_{x \to 2} \frac{x^3 - 8}{x - 2} = \frac{2^3 - 8}{2 - 2} = \frac{8 - 8}{2 - 2} = \frac{0}{0}$ $\left(\frac{0}{0} \text{ form}\right)$

$\therefore \lim_{x \to 2} \frac{x^3 - 8}{x - 2} = \lim_{x \to 2} \frac{x^3 - 2^3}{x - 2}$

By factorization method

$= \lim_{x \to 2} \frac{(x - 2)\left(x^2 + 2^2 + 2x\right)}{x - 2}$

$= \lim_{x \to 2} \left(x^2 + 2^2 + 2x\right)$

$= 2^2 + 2^2 + 2(2)$

$= 4 + 4 + 4 = 12$

Example 6. Evaluate $\lim_{x \to 0} \frac{\sqrt{a + x} - \sqrt{a - x}}{x}$.

Sol. $\lim_{x \to 0} \frac{\sqrt{a + x} - \sqrt{a - x}}{x} = \frac{\sqrt{a + 0} - \sqrt{a - 0}}{0} = \frac{\sqrt{a} - \sqrt{a}}{0} = \frac{0}{0}$ $\left(\frac{0}{0} \text{ form}\right)$

$\therefore \lim_{x \to 0} \frac{\sqrt{a + x} - \sqrt{a - x}}{x}$

By rationalization method

$= \lim_{x \to 0} \frac{\sqrt{a + x} - \sqrt{a - x}}{x} \times \frac{\sqrt{a + x} + \sqrt{a - x}}{\sqrt{a + x} + \sqrt{a - x}}$

$$= \lim_{x \to 0} \frac{\left(\sqrt{a+x}\right)^2 - \left(\sqrt{a-x}\right)^2}{x\left(\sqrt{a+x}+\sqrt{a-x}\right)}$$

$$= \lim_{x \to 0} \frac{a+x-a+x}{x\left(\sqrt{a+x}+\sqrt{a-x}\right)}$$

$$= \lim_{x \to 0} \frac{2x}{x\left(\sqrt{a+x}+\sqrt{a-x}\right)}$$

$$= \lim_{x \to 0} \frac{2}{\left(\sqrt{a+x}+\sqrt{a-x}\right)}$$

$$= \frac{2}{\left(\sqrt{a+0}+\sqrt{a-0}\right)}$$

$$= \frac{2}{2\sqrt{a}} = \frac{1}{\sqrt{a}}$$

Method of evaluation of algebraic limits when $x \to \infty$ **:**

Example 7. Evaluate $\lim_{x \to \infty} \frac{(x+1)(x+2)}{(x+3)(x+4)}$.

Sol. $\lim_{x \to \infty} \frac{(x+1)(x+2)}{(x+3)(x+4)} = \frac{(\infty+1)(\infty+2)}{(\infty+3)(\infty+4)} = \frac{\infty}{\infty}$ $\left(\frac{\infty}{\infty} \text{ form}\right)$

$$\therefore \lim_{x \to \infty} \frac{(x+1)(x+2)}{(x+3)(x+4)} = \lim_{x \to \infty} \frac{x\left(1+\frac{1}{x}\right)x\left(1+\frac{2}{x}\right)}{x\left(1+\frac{3}{x}\right)x\left(1+\frac{4}{x}\right)}$$

$$= \lim_{x \to \infty} \frac{x^2\left(1+\frac{1}{x}\right)\left(1+\frac{2}{x}\right)}{x^2\left(1+\frac{3}{x}\right)\left(1+\frac{4}{x}\right)} = \lim_{x \to \infty} \frac{\left(1+\frac{1}{x}\right)\left(1+\frac{2}{x}\right)}{\left(1+\frac{3}{x}\right)\left(1+\frac{4}{x}\right)}$$

$$= \frac{\left(1+\frac{1}{\infty}\right)\left(1+\frac{2}{\infty}\right)}{\left(1+\frac{3}{\infty}\right)\left(1+\frac{4}{\infty}\right)} = \frac{(1+0)(1+0)}{(1+0)(1+0)} = \frac{1}{1} = 1$$

Example 8. Evaluate $\lim_{x \to \infty} \frac{\left(x^2-1\right)}{\left(x^2+2x+1\right)(x+5)}$.

Sol. $\lim_{x \to \infty} \frac{\left(x^2-1\right)}{\left(x^2+2x+1\right)(x+5)} = \frac{\left(\infty^2-1\right)}{\left(\infty^2+2(\infty)+1\right)(\infty+5)} = \frac{\infty}{\infty}$ $\left(\frac{\infty}{\infty} \text{ form}\right)$

$$\therefore \lim_{x \to \infty} \frac{\left(x^2-1\right)}{\left(x^2+2x+1\right)(x+5)} = \lim_{x \to \infty} \frac{x^2\left(1-\frac{1}{x^2}\right)}{x^2\left(1+\frac{2}{x}+\frac{1}{x^2}\right)x\left(1+\frac{5}{x}\right)}$$

$$= \lim_{x\to\infty} \frac{\left(1-\frac{1}{x^2}\right)}{x\left(1+\frac{2}{x}+\frac{1}{x^2}\right)\left(1+\frac{5}{x}\right)}$$

$$= \frac{\left(1-\frac{1}{\infty^2}\right)}{\infty\left(1+\frac{2}{\infty}+\frac{1}{\infty^2}\right)\left(1+\frac{5}{\infty}\right)} = \frac{(1-0)}{\infty(1+0+0)(1+0)} = \frac{1}{\infty} = 0$$

Example 9. Evaluate $\lim_{x\to\frac{\pi}{2}} (\sin x - \cos x)$.

Sol. $\lim_{x\to\frac{\pi}{2}} (\sin x - \cos x) = \sin\frac{\pi}{2} - \cos\frac{\pi}{2} = 1 - 0 = 1$

Example 10. Evaluate $\lim_{x\to 0} \frac{\sin 5x}{6x}$.

Sol. $\lim_{x\to 0} \frac{\sin 5x}{6x} = \frac{\sin(5\times 0)}{6\times 0} = \frac{\sin(0)}{0} = \frac{0}{0}$ $\left(\frac{0}{0} \text{ form}\right)$

$\therefore \lim_{x\to 0} \frac{\sin 5x}{6x} = \lim_{x\to 0} \frac{\sin 5x}{5x} \times \frac{5}{6} = 1\times\frac{5}{6} = \frac{5}{6}$ $\left[\text{by } \lim_{x\to 0}\frac{\sin x}{x} = 1\right]$

Example 11. Evaluate $\lim_{x\to 0} \frac{9x}{\tan 3x}$.

Sol. $\lim_{x\to 0} \frac{9x}{\tan 3x} = \frac{9\times 0}{\tan(3\times 0)} = \frac{0}{\tan(0)} = \frac{0}{0}$ $\left(\frac{0}{0} \text{ form}\right)$

$\therefore \lim_{x\to 0} \frac{9x}{\tan 3x} = \lim_{x\to 0} \frac{9x}{\frac{\tan 3x}{3x}\times 3x} = \lim_{x\to 0} \frac{9x}{1\times 3x} = \frac{9}{3} = 3$ $\left[\text{by } \lim_{x\to 0}\frac{\tan x}{x} = 1\right]$

Examples based on trigonometric formulas:

Example 12. Evaluate $\lim_{x\to 0} \frac{\sin 4x - \sin 2x}{\sin 7x + \sin 3x}$.

Sol. $\lim_{x\to 0} \frac{\sin 4x - \sin 2x}{\sin 7x + \sin 3x} = \frac{\sin 0 - \sin 0}{\sin 0 + \sin 0} = \frac{0-0}{0+0} = \frac{0}{0}$ $\left(\frac{0}{0} \text{ form}\right)$

$$\therefore \lim_{x\to 0} \frac{\sin 4x - \sin 2x}{\sin 7x + \sin 3x} = \lim_{x\to 0} \frac{2\cos\left(\frac{4x+2x}{2}\right)\sin\left(\frac{4x-2x}{2}\right)}{2\sin\left(\frac{7x+3x}{2}\right)\cos\left(\frac{7x-3x}{2}\right)}$$

$$= \lim_{x\to 0} \frac{\cos 3x \sin x}{\sin 5x \cos 2x} = \lim_{x\to 0} \frac{\cos 3x \times \frac{\sin x}{x} \times x}{\frac{\sin 5x}{5x} \times 5x \times \cos 2x}$$

$$= \lim_{x\to 0} \frac{\cos 0 \times 1 \times x}{1\times 5x \times \cos 0} = \lim_{x\to 0} \frac{1\times 1}{1\times 5\times 1} = \frac{1}{5}$$

Examples based on $\lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a$ **:**

Example 13. Evaluate $\lim_{x \to 0} \frac{2^x - 1}{x}$.

Sol. Applying the above formula, we get

$\therefore \lim_{x \to 0} \frac{2^x - 1}{x} = \log_e 2$ $\qquad$ $(here\ a = 2)$

Example 14. Evaluate $\lim_{x \to 0} \frac{4^x - 3^x}{\sin x}$.

Sol. $\therefore \lim_{x \to 0} \frac{4^x - 3^x}{\sin x} = \lim_{x \to 0} \frac{4^x - 1 + 1 - 3^x}{\frac{\sin x}{x} \times x} = \lim_{x \to 0} \frac{(4^x - 1) - (3^x - 1)}{1 \times x}$

$= \lim_{x \to 0} \left[\left(\frac{4^x - 1}{x} \right) - \left(\frac{3^x - 1}{x} \right) \right]$

$= \log_e 4 - \log_e 3 = \log_e \frac{4}{3}$ $\qquad$ $\left(\log_e a - \log_e b = \log_e \frac{a}{b} \right)$

Example 15. Evaluate $\lim_{x \to 0} \frac{e^{x^2} - 1}{x \tan x}$.

Sol. $\therefore \lim_{x \to 0} \frac{e^{x^2} - 1}{x \tan x} = \lim_{x \to 0} \left[\frac{e^{x^2} - 1}{x^2} \times \frac{x^2}{x \tan x} \right]$

$= \log_e e \times \lim_{x \to 0} \frac{x^2}{x \tan x} = 1 \times \lim_{x \to 0} \frac{x}{\tan x} = 1$ $\qquad$ $(\log_e e = 1)$

EXERCISE-I

1. If $f(x) = x^3 + 2x^2 - 3x + 1$, find $f(-1)$.
2. If $f(x) = x^2 + x + 1$, find $f(2).f(3)$.
3. The limit of $f(x) = \frac{cosx}{x}$ as $x \to 0$ is ....
   a. 1 b. 0 c. -1 d. Does not exist
4. Evaluate $\lim_{x \to 2} \frac{x^3 - 8}{x + 2}$.
5. Evaluate $\lim_{x \to 5} \frac{x^2 - 25}{x - 5}$.
6. Evaluate $\lim_{x \to 3} \frac{x - 3}{x^2 - 9}$.
7. Evaluate $\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - 4x + 3}$.
8. Evaluate $\lim_{x \to 3} \frac{\sqrt{x} - \sqrt{3}}{x - 3}$.
9. Evaluate $\lim_{x \to 0} \frac{\sqrt{2 + x} - \sqrt{2 - x}}{2x}$.
10. Evaluate $\lim_{x \to 4} \frac{x^3 - 64}{x - 4}$.

11. Evaluate $\lim_{x \to 2} \frac{\sqrt{x}-\sqrt{2}}{x-2}$.

12. Evaluate $\lim_{x \to 5} \frac{x^{\frac{1}{3}}-5^{\frac{1}{3}}}{x-5}$.

13. Evaluate $\lim_{x \to \infty} \frac{x(x-5)}{x^2+5}$.

14. Evaluate $\lim_{x \to \infty} \frac{(2x+5)(x+3)}{(3x^2+2x-1)(x+9)}$.

15. Evaluate $\lim_{x \to \infty} \frac{(x^5+x^3+5)}{(x^2-9)(x+8)}$.

16. Evaluate $\lim_{x \to \infty} \frac{(x^7+2x^2-6)}{(x^3+5)(x^2+x+1)}$.

17. Evaluate $\lim_{x \to 0} \frac{4x}{\sin 2x}$.

18. Evaluate $\lim_{x \to 0} \frac{\tan 6x}{3x}$.

19. Evaluate $\lim_{x \to 0} \frac{\tan 4x}{\sin 2x}$.

20. Evaluate $\lim_{x \to 0} \frac{\sin 3x - \tan 2x}{x}$.

21. Evaluate $\lim_{x \to 0} \frac{\tan 4x + 2x}{2x}$.

22. Evaluate $\lim_{x \to 0} \frac{\sin 6x + \sin 4x}{\sin 5x - \sin 3x}$.

23. Evaluate $\lim_{x \to 0} \frac{\sin 4x - x}{6x - \sin 3x}$.

24. Evaluate $\lim_{x \to 0} \frac{5^x - 1}{x}$.

25. Evaluate $\lim_{x \to 0} \frac{3^x - 2^x}{x}$.

26. Evaluate $\lim_{x \to 0} \frac{5^x - 3^x}{\tan x}$.

27. Evaluate $\lim_{x \to 0} \frac{a^x - 1}{b^x - 1}$.

28. Evaluate $\lim_{x \to 0} \frac{2^x - 1}{3^x - 1}$.

29. Evaluate $\lim_{x \to 0} \frac{a^{\tan x} - 1}{\tan x}$.

30. Evaluate $\lim_{x \to 0} \frac{e^{\sin x} - 1}{x}$.

ANSWERS

1. 5   2. 91   3. d   4. 0

5. 10	6. $\frac{1}{6}$	7. $-\frac{3}{2}$	8. $\frac{1}{2\sqrt{3}}$
9. $\frac{1}{2\sqrt{2}}$	10. 48	11. $\frac{1}{2\sqrt{2}}$	12. $\frac{1}{3(5)^{2/3}}$
13. 1	14. 0	15.	16.
17. 2	18. 2	19. 2	20. 1
21. 3	22. 5	23. 1	24.
25.	26.	27.	28.
29.	30. 1		

4.2 DIFFERENTIATION

Increment: Increment is the quantity by which the value of a variable changes. It may be positive or negative. e.g. suppose the value of a variable x changes from 5 to 5.3 then 0.3 is the increment in x. Similarly, if the value of variable x changes from 5 to 4.5 then -0.5 is the increment in x.

Usually δx represents the increment in x, δy represents the increment in y, δz represents the increment in z etc.

Derivative or Differential Co-efficient: If y is a function of x. Let δx be the increment in x and δy be the corresponding increment in y, then $\lim_{\delta x \to 0} \frac{\delta y}{\delta x}$ (*if it exists*) is called the derivative or differential co-efficient of y with respect to x and is dented by $\frac{dy}{dx}$.

i.e. $$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

Differentiation:

Let $y = f(x)$ (1)

Let δx be the increment in x and δy be the corresponding increment in y, then

$y + \delta y = f(x + \delta x)$ (2)

Subtracting equation (1) from equation (2), we get $y + \delta y - y = f(x + \delta x) - f(x)$

$\Rightarrow \quad \delta y = f(x + \delta x) - f(x)$

Dividing both sides by δx, we get

$$\frac{\delta y}{\delta x} = \frac{f(x + \delta x) - f(x)}{\delta x}$$

Taking limit $\delta x \to 0$ on both sides, we get

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

If this limit exists, we write it as

$$\frac{dy}{dx} = f'(x)$$

where $f'(x)=\lim_{\delta x\to 0}\dfrac{f(x+\delta x)-f(x)}{\delta x}$.

This is called the differentiation or derivative of the function $f(x)$ with respect to x .

Notations: The first order derivative of the function $f(x)$ with respect to x can be represented in the following ways:

$$\frac{d}{dx}(f(x)),\ \frac{df}{dx},\ f'(x),\ f_1(x) \text{ etc.}$$

Similarly, the first order derivative of y with respect to x can be represented as:

$$\frac{dy}{dx}, y', y_1 \text{ etc.}$$

Physical Interpretation of Derivatives:

Let the variable t represents the time and the function $f(t)$ represents the distance travelled in time t.

We know that $Speed = \dfrac{Distance\ travelled}{Time\ taken}$

If time interval is between ' a ' & ' $a+h$ ' . Here h be increment in a . Then the speed in that interval is given by

$$\frac{f(a+h)-f(a)}{a+h-a}=\frac{f(a+h)-f(a)}{h}$$

If we take $h\to 0$ then $\dfrac{f(a+h)-f(a)}{h}$ approaches the speed at time $t=a$. Thus we can say that derivative is related in the similar way as speed is related to the distance travelled by a moving particle.

Some Properties of Differentiation:

If $f(x)$ and $g(x)$ are differentiable functions, then

(i) $\dfrac{d}{dx}(K)=0$ where K is some constant.

(ii) $\dfrac{d}{dx}(K.f(x))=K.\dfrac{d}{dx}(f(x))$ where K is some constant.

(iii) $\dfrac{d}{dx}[f(x)+g(x)]=\dfrac{d}{dx}(f(x))+\dfrac{d}{dx}(g(x))$

(iv) $\dfrac{d}{dx}[f(x)-g(x)]=\dfrac{d}{dx}(f(x))-\dfrac{d}{dx}(g(x))$

(v) $\dfrac{d}{dx}[f(x).g(x)]=f(x).\dfrac{d}{dx}(g(x))+g(x).\dfrac{d}{dx}(f(x))$

This property is known as Product Rule of differentiation.

(vi) $\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right]=\dfrac{g(x)\dfrac{d}{dx}(f(x))-f(x)\dfrac{d}{dx}(g(x))}{(g(x))^2}$ provided that $g(x)\neq 0$

This property is known as Quotient Rule of differentiation.

Differentiation of standard functions:

(i) $\frac{d}{dx}\left(x^n\right) = n x^{n-1}$. This is known as power formula, here n is any real number.

(ii) $\frac{d}{dx}(\sin x) = \cos x$

(iii) $\frac{d}{dx}(\cos x) = -\sin x$

(iv) $\frac{d}{dx}(\tan x) = \sec^2 x$

(v) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(vi) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

(vii) $\frac{d}{dx}\left(a^x\right) = a^x \log_e a \quad \text{here } a > 0 \ \& \ a \neq 1$

(viii) $\frac{d}{dx}\left(e^x\right) = e^x \log_e e = e^x$

(ix) $\frac{d}{dx}(\log_e x) = \frac{1}{x}$

(x) $\frac{d}{dx}(\log_a x) = \frac{1}{x}\log_a e$

(xi) $\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$

(xii) $\frac{d}{dx}\left(\cos^{-1} x\right) = \frac{-1}{\sqrt{1-x^2}}$

(xiii) $\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$

(xiv) $\frac{d}{dx}\left(\cot^{-1} x\right) = \frac{-1}{1+x^2}$

(xv) $\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{|x|\sqrt{x^2-1}}$

(xvi) $\frac{d}{dx}\left(\operatorname{cosec}^{-1} x\right) = \frac{-1}{|x|\sqrt{x^2-1}}$

Example 16. Differentiate $y = x^{10}$ with respect to x .

Sol. Given that $y = x^{10}$

Differentiating it with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^{10}\right) = 10x^9$$

4.3 DIFFERENTIATION OF SUM, PRODUCT AND QUOTIENT OF FUNCTIONS Differentiation of sum of two or more functions, product of two or more functions and quotient of two or more functions are explained with following examples as:

Example 17. Differentiate $y = 5 - x^6$ with respect to x .

Sol. Given that $y = 5 - x^6$

Differentiating it with respect to x , we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(5-x^{6}\right)=\frac{d}{dx}(5)-\frac{d}{dx}\left(x^{6}\right)$$

$$=0-6x^{5}=-6x^{5}$$

Example 18. Differentiate $y=\sqrt{x}-\frac{1}{\sqrt{x}}$ with respect to x.

Sol. Given that $y=\sqrt{x}-\frac{1}{\sqrt{x}}$

Differentiating it with respect to x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)=\frac{d}{dx}\left(\sqrt{x}\right)-\frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right)$$

$$=\frac{d}{dx}\left(x^{\frac{1}{2}}\right)-\frac{d}{dx}\left(x^{-\frac{1}{2}}\right)=\frac{1}{2}x^{\frac{1}{2}-1}-\left(-\frac{1}{2}\right)x^{-\frac{1}{2}-1}$$

$$=\frac{1}{2}x^{-\frac{1}{2}}+\frac{1}{2}x^{-\frac{3}{2}}$$

Example 19. Differentiate $y=\sin x-e^{x}+2^{x}$ with respect to x.

Sol. Given that $y=\sin x-e^{x}+2^{x}$

Differentiating it with respect to x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(\sin x-e^{x}+2^{x}\right)=\frac{d}{dx}(\sin x)-\frac{d}{dx}\left(e^{x}\right)+\frac{d}{dx}\left(2^{x}\right)$$

$$=\cos x-e^{x}+2^{x}\log_{e}2$$

Example 20. Differentiate $y=e^{x}.a^{x}+2x^{3}-\log x$ with respect to x.

Sol. Given that $y=e^{x}.a^{x}+2x^{3}-\log x=(ea)^{x}+2x^{3}-\log x$

Differentiating it with respect to x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left((ea)^{x}+2x^{3}-\log x\right)=\frac{d}{dx}(ea)^{x}+2\frac{d}{dx}\left(x^{3}\right)-\frac{d}{dx}(\log x)$$

$$=(ea)^{x}\log_{e}(ea)+2\left(3x^{2}\right)-\frac{1}{x}=(ea)^{x}\log_{e}(ea)+6x^{2}-\frac{1}{x}$$

__Chain Rule__: If $f(x)$ and $g(x)$ are two differentiable functions then

$$\frac{d}{dx}(f(g(x)))=f'(g(x)).\frac{d}{dx}(g(x))=f'(g(x)).\,g'(x)$$

So, we may generalize our basic formulas as:

(i) $\frac{d}{dx}\left((f(x))^{n}\right)=n\,(f(x))^{n-1}.\,f'(x)$ here n is any real number.

(ii) $\frac{d}{dx}(\sin(f(x)))=\cos(f(x)).\,f'(x)$ etc.

__**Examples based on Chain Rule:**__

Example 21. Differentiate $y=\sin(2x+1)$ with respect to x.

Sol. Given that $y=\sin(2x+1)$

Differentiating it with respect to x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\sin(2x+1)=\cos(2x+1).\frac{d}{dx}(2x+1)$$

$$=\cos(2x+1).(2\times1+0)=2\cos(2x+1)$$

Example 22. Differentiate $y = \tan(\cos x)$ with respect to x.

Sol. Given that $y = \tan(\cos x)$

Differentiating it with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\tan(\cos x) = \sec^2(\cos x).\frac{d}{dx}(\cos x)$$

$$= \sec^2(\cos x).(-\sin x) = -\sin x.\sec^2(\cos x)$$

Example 23. Differentiate $r = \sin(\log(5s))$ with respect to s.

Sol. Given that $r = \sin(\log(5s))$

Differentiating it with respect to s, we get

$$\frac{dr}{ds} = \frac{d}{ds}(\sin(\log(5s))) = \cos(\log(5s)).\frac{d}{ds}(\log(5s))$$

$$= \cos(\log(5s)).\frac{1}{5s}.\frac{d}{ds}(5s) = \cos(\log(5s)).\frac{1}{5s}\times 5 = \frac{\cos(\log(5s))}{5}$$

Examples based on Product Rule:

Example 24. Differentiate $y = x\cos x$ with respect to x.

Sol. Given that $y = x\cos x$

Differentiating it with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(x\cos x) = x.\frac{d}{dx}(\cos x) + \cos x\frac{dx}{dx}$$

$$= x.(-\sin x) + \cos x.1 = -x\sin x + \cos x$$

Example 25. Differentiate $y = \log x\tan x$ with respect to x.

Sol. Given that $y = \log x\tan x$

Differentiating it with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(\log x\tan x) = \log x.\frac{d}{dx}(\tan x) + \tan x\frac{d}{dx}(\log x)$$

$$= \log x \times (\sec^2 x) + \tan x \times \frac{1}{x} = \log x.\sec^2 x + \frac{\tan x}{x}$$

Examples based on Quotient Rule:

Example 26. Differentiate $y = \frac{\sin x}{x}$ with respect to x.

Sol. Given that $y = \frac{\sin x}{x}$

Differentiating it with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\sin x}{x}\right) = \frac{x.\frac{d}{dx}(\sin x) - \sin x.\frac{dx}{dx}}{x^2}$$

$$= \frac{x.\cos x - \sin x.1}{x^2} = \frac{x\cos x - \sin x}{x^2}$$

Example 27. Differentiate $y = \frac{x}{\sqrt{1+x^2}}$ with respect to x.

Sol. Given that $y = \frac{x}{\sqrt{1+x^2}}$

Differentiating it with respect to x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{x}{\sqrt{1+x^2}}\right)=\frac{\sqrt{1+x^2}\cdot\frac{dx}{dx}-x\cdot\frac{d}{dx}\left(\sqrt{1+x^2}\right)}{\left(\sqrt{1+x^2}\right)^2}$$

$$=\frac{\sqrt{1+x^2}\cdot 1-x\cdot\frac{1}{2}\left(1+x^2\right)^{-\frac{1}{2}}\frac{d}{dx}\left(1+x^2\right)}{\left(1+x^2\right)}$$

$$=\frac{\left(\sqrt{1+x^2}-\frac{x}{2\sqrt{1+x^2}}\times 2x\right)}{\left(1+x^2\right)}=\frac{\left(\sqrt{1+x^2}-\frac{x^2}{\sqrt{1+x^2}}\right)}{\left(1+x^2\right)}$$

$$=\frac{\left(\frac{\left(\sqrt{1+x^2}\right)^2-x^2}{\sqrt{1+x^2}}\right)}{\left(1+x^2\right)}$$

$$=\frac{1+x^2-x^2}{\left(1+x^2\right)\sqrt{1+x^2}}=\frac{1}{\left(1+x^2\right)\sqrt{1+x^2}}$$

Examples based on Parametric Form:

Example 28. Evaluate $\frac{dy}{dx}$ if $x=t^2$ *and* $y=2t$.

Sol. Given that $x=t^2$ *and* $y=2t$

Differentiating x with respect to t, we get

$$\frac{dx}{dt}=\frac{d}{dt}\left(t^2\right)=2t$$

Differentiating y with respect to t, we get

$$\frac{dy}{dt}=\frac{d}{dt}(2t)=2$$

$$\therefore\ \frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{2}{2t}=\frac{1}{t}\ \text{or}\ \frac{1}{\sqrt{x}}$$

Example 29. Evaluate $\frac{dy}{dx}$ if $x=\cos 4\theta$ *and* $y=\sin 2\theta$.

Sol. Given that $x=\cos 4\theta$ *and* $y=\sin 2\theta$

Differentiating x with respect to θ, we get

$$\frac{dx}{d\theta}=\frac{d}{d\theta}\left(\cos 4\theta\right)=-\sin 4\theta\times\frac{d}{d\theta}\left(4\theta\right)=-4\sin 4\theta$$

Differentiating y with respect to θ, we get

$$\frac{dy}{d\theta}=\frac{d}{d\theta}\left(\sin 2\theta\right)=\cos 2\theta\times\frac{d}{d\theta}\left(2\theta\right)=2\cos 2\theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{2\cos 2\theta}{-4\sin 4\theta} = -\frac{\cos 2\theta}{2\sin 4\theta}$$

$$\log y = \log x^x$$

$$\log y = x \log x \qquad \left(\log a^b = b \log a\right)$$

Differentiating it with respect to x, we get

$$\frac{d}{dx}(\log y) = \frac{d}{dx}(x \log x)$$

$$\frac{1}{y}\frac{dy}{dx} = x\frac{d}{dx}(\log x) + \log x \frac{dx}{dx}$$

$$\frac{1}{y}\frac{dy}{dx} = x . \frac{1}{x} + \log x . 1$$

<u>Logarithmic Differentiation</u> :

Let $f(x)$ and $g(x)$ are two differentiable function and $y = f(x)^{g(x)}$
To differentiate y, first we take logarithm of y :

$$\log y = \log\left(f(x)^{g(x)}\right)$$

$$\log y = g(x)\log(f(x)) \qquad \left(\log a^b = b \log a\right)$$

Differentiating it with respect to x, we get

$$\frac{d}{dx}(\log y) = \frac{d}{dx}\left(g(x)\log(f(x))\right)$$

$$\frac{1}{y}\frac{dy}{dx} = g(x)\frac{d}{dx}(\log f(x)) + \log f(x)\frac{d}{dx}(g(x))$$

$$\frac{1}{y}\frac{dy}{dx} = g(x)\frac{1}{f(x)} . f'(x) + \log f(x) g'(x)$$

$$\frac{dy}{dx} = y\left[\frac{g(x)}{f(x)} f'(x) + \log f(x) g'(x)\right]$$

$$\frac{dy}{dx} = f(x)^{g(x)}\left[\frac{g(x)}{f(x)} f'(x) + \log f(x) g'(x)\right]$$

<u>Examples based on Derivative of Logarithmic Differentiation</u> :

Example 30. Differentiate $y = x^x$ with respect to x.

Sol. Given that $y = x^x$

Taking logarithm on both sides, we get

$$\frac{1}{y}\frac{dy}{dx} = 1 + \log x$$

$$\frac{dy}{dx} = y(1 + \log x)$$

Example 31. Differentiate $y = x^{\sin x}$ with respect to x.

Sol. Given that $y = x^{\sin x}$

Taking logarithm on both sides, we get

$$\log y = \log x^{\sin x}$$

$$\log y = \sin x \,.\, \log x \qquad \left(\log a^{b} = b \log a\right)$$

Differentiating it with respect to x, we get

$$\frac{d}{dx}(\log y) = \frac{d}{dx}(\sin x \,.\log x)$$

$$\frac{1}{y}\frac{dy}{dx} = \sin x \frac{d}{dx}(\log x) + \log x \frac{d}{dx}(\sin x)$$

$$\frac{1}{y}\frac{dy}{dx} = \sin x \,.\, \frac{1}{x} + \log x \,.\cos x$$

$$\frac{dy}{dx} = y\left(\frac{\sin x}{x} + \log x \cos x\right)$$

$$\frac{dy}{dx} = x^{\sin x}\left(\frac{\sin x}{x} + \log x \cos x\right)$$

Example 32. Differentiate $y = (\sin x)^{\cos x}$ with respect to x.

Sol. Given that $y = (\sin x)^{\cos x}$

Taking logarithm on both sides, we get

$$\log y = \log (\sin x)^{\cos x}$$

$$\log y = \cos x \log (\sin x) \qquad \left(\log a^{b} = b \log a\right)$$

Differentiating it with respect to x, we get

$$\frac{d}{dx}(\log y) = \frac{d}{dx}(\cos x \log (\sin x))$$

$$\frac{1}{y}\frac{dy}{dx} = \cos x \frac{d}{dx}(\log (\sin x)) + \log (\sin x) \frac{d}{dx}(\cos x)$$

$$\frac{1}{y}\frac{dy}{dx} = \cos x \frac{1}{\sin x}\frac{d}{dx}(\sin x) + \log (\sin x).(-\sin x)$$

$$\frac{1}{y}\frac{dy}{dx} = \cot x \cos x - \sin x \log (\sin x)$$

$$\frac{dy}{dx} = y\left(\cot x \cos x - \sin x \log (\sin x)\right)$$

$$\frac{dy}{dx} = (\sin x)^{\cos x}\left(\cot x \cos x - \sin x \log (\sin x)\right)$$

Examples based on Derivative of Infinite Series form :

Example 33. Differentiate $\sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \ldots}}}}$ with respect to x.

Sol. Let $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \ldots}}}}$

$$y = \sqrt{\sin x + y}$$

$$y^{2} = \sin x + y$$

Differentiating it with respect to x, we get

$$\frac{d}{dx}(y^{2}) = \frac{d}{dx}(\sin x + y)$$

$$2y\frac{dy}{dx}-\frac{dy}{dx}=\cos x$$

$$\frac{dy}{dx}=\frac{\cos x}{(2y-1)}$$

Example 34. Differentiate with respect to x.

Sol. Let

$$y=x^y$$

Taking logarithm on both sides, we get

$$\log y=\log x^y$$

$$\log y=y\log x$$

Differentiating it with respect to x, we get

$$\frac{d}{dx}(\log y)=\frac{d}{dx}(y\log x)$$

$$\frac{1}{y}\frac{dy}{dx}=y\frac{d}{dx}(\log x)+\log x\frac{dy}{dx}$$

$$\frac{1}{y}\frac{dy}{dx}-\log x\frac{dy}{dx}=\frac{y}{x}$$

$$\left(\frac{1}{y}-\log x\right)\frac{dy}{dx}=\frac{y}{x}$$

$$\left(\frac{1-y\log x}{y}\right)\frac{dy}{dx}=\frac{y}{x}$$

$$\frac{dy}{dx}=\frac{y^2}{x(1-y\log x)}$$

4.4 SUCCESSIVE DIFFERENTIATION OR HIGHER ORDER DERIVATIVE

Let $y=f(x)$ be a differentiable function, then $\frac{dy}{dx}$ represents the first order derivative of y with respect to x. If we may further differentiate it i.e. $\frac{d}{dx}\left(\frac{dy}{dx}\right)$, then it is called second order derivative of y with respect to x. Some other way to represent second order derivative of y with respect to x: $\frac{d^2y}{dx^2}$, y'', $\dot{y}_2$

So, successive derivatives of y with respect to x can be represented as

$$\frac{dy}{dx},\ \frac{d^2y}{dx^2},\ \frac{d^3y}{dx^3},\ \ldots,\ \frac{d^ny}{dx^n}\ etc.$$

Example 35. If $y=x^8-12x^5+5x^3-12$, find $\frac{d^2y}{dx^2}$.

Sol. Given that $y = x^8 - 12x^5 + 5x^3 - 12$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^8 - 12x^5 + 5x^3 - 12\right)$$

$\Rightarrow \quad \frac{dy}{dx} = 8x^7 - 60x^4 + 15x^2$

Again differentiating with respect to x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(8x^7 - 60x^4 + 15x^2\right)$$

$\Rightarrow \quad \frac{d^2y}{dx^2} = 56x^6 - 240x^3 + 30$

Example 36. If $f(x) = x^2 .\sin x$, find $f'(0)$ and $f''\left(\frac{\pi}{2}\right)$.

Sol. Given that $f(x) = x^2 .\sin x$

Differentiating with respect to x, we get

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}\left(x^2 .\sin x\right)$$

$\Rightarrow \quad f'(x) = x^2 \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}\left(x^2\right)$

$\Rightarrow \quad f'(x) = x^2 \cos x + \sin x \,.\, 2x$

$\Rightarrow \quad f'(x) = x^2 \cos x + 2x \sin x \qquad (1)$

Again differentiating with respect to x, we get

$$f''(x) = \frac{d}{dx}\left(x^2 \cos x + 2x \sin x\right)$$

$\Rightarrow \quad f''(x) = \frac{d}{dx}\left(x^2 \cos x\right) + 2\frac{d}{dx}(x \sin x)$

$\Rightarrow \quad f''(x) = x^2 \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}\left(x^2\right) + 2\left[x \frac{d}{dx}(\sin x) + \sin x \frac{dx}{dx}\right]$

$\Rightarrow \quad f''(x) = -x^2 \sin x + 2x \cos x + 2\left[x \cos x + \sin x\right]$

$\Rightarrow \quad f''(x) = -x^2 \sin x + 4x \cos x + 2 \sin x \qquad (2)$

Put $x = 0$ in equation (1), we get

$$f'(0) = (0)^2 \cos 0 + 2\times 0 \times \sin 0 = 0$$

Put $x = \frac{\pi}{2}$ in equation (2), we get

$$f''\left(\frac{\pi}{2}\right) = -\left(\frac{\pi}{2}\right)^2 \sin\left(\frac{\pi}{2}\right) + 4\times\frac{\pi}{2}\times\cos\left(\frac{\pi}{2}\right) + 2\sin\left(\frac{\pi}{2}\right)$$

$\Rightarrow \quad f''\left(\frac{\pi}{2}\right) = -\frac{\pi^2}{4} + 2$

Example 37. If $y = \sin Ax + \cos Ax$, prove that $\frac{d^2y}{dx^2} + A^2 y = 0$.

Soln. Given that $y = \sin Ax + \cos Ax$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(\sin Ax + \cos Ax)$$

$\Rightarrow \quad \frac{dy}{dx} = A \cos Ax - A \sin Ax$

Again differentiating with respect to x, we get

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}(A\cos Ax - A\sin Ax)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2}=-A^2\sin Ax - A^2\cos Ax$$

$$\Rightarrow \quad \frac{d^2y}{dx^2}=-A^2(\sin Ax+\cos Ax)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2}=-A^2\,y$$

$$\Rightarrow \quad \frac{d^2y}{dx^2}+A^2\,y=0$$

Example 38. If $y=e^{-Ax}$, prove that $\frac{d^2y}{dx^2}+A\frac{dy}{dx}=0$.

Soln. Given that $y=e^{-Ax}$

Differentiating with respect to x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(e^{-Ax}\right)$$

$$\Rightarrow \quad \frac{dy}{dx}=e^{-Ax}\frac{d}{dx}(-Ax)$$

$$\Rightarrow \quad \frac{dy}{dx}=-A\,e^{-Ax}$$

$$\Rightarrow \quad \frac{dy}{dx}=-A\,y$$

Again differentiating with respect to x, we get

$$\frac{d^2y}{dx^2}=-A\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2}+A\frac{dy}{dx}=0$$

EXERCISE-II

1. Differentiate $y=\sqrt{x}$ with respect to x.
2. Differentiate $y=x^{-\frac{5}{2}}$ with respect to x.
3. Differentiate $y=2-x+3x^2$ with respect to x.
4. Differentiate $y=(x+3)(x-1)$ with respect to x.
5. Differentiate $y=2\log x-5\sec x$ with respect to x.
6. Differentiate $y=\frac{x^2+7}{x}$ with respect to x.
7. Differentiate $y=\cos\left(x^2+x+1\right)$ with respect to x.
8. Differentiate $y=\sin^3 x$ with respect to x.
9. Differentiate $y=\cos(\sin x)$ with respect to x.
10. Differentiate $y=\log(\tan x)$ with respect to x.
11. Differentiate $v=e^{5t^2}$ with respect to t.
12. Differentiate $z=2^{s^2+9}$ with respect to s.

13. Differentiate $y = x^2 \sin x$ with respect to x.
14. Differentiate $y = \cos x \log x$ with respect to x.
15. Differentiate $y = \left(3t^2\right)2^t$ with respect to t .–9
16. Differentiate $y = \frac{\log(7x)}{x+1}$ with respect to x.
17. Differentiate $y = \frac{\log x}{\tan x}$ with respect to x.
18. Differentiate $y = \frac{x^2+1}{\sin x}$ with respect to x.
19. Differentiate $y = \frac{\tan 2x}{e^{2x}}$ with respect to x.
20. Evaluate $\frac{dy}{dx}$ if $x = 2t^2 + 1$ and $y = t^3$.
21. Evaluate $\frac{dy}{dx}$ if $x = \log 2t$ and $y = 2\tan t$.
22. Evaluate $\frac{dy}{dx}$ if $x = \sec t$ and $y = 5^t + 7t - 11$.
23. Differentiate $y = (\cos x)^x$ with respect to x.
24. Differentiate $y = x^{\cos x}$ with respect to x.
25. Differentiate $\sqrt{x+\sqrt{x+\sqrt{x+\sqrt{x+\ldots}}}}$ with respect to x.
26. Differentiate $\cos x^{\cos x^{\cos x \ldots}}$ with respect to x.
27. If $y = \log(\sin x) + e^{5x}$, find $\frac{d^2y}{dx^2}$.
28. If $y = x^3 . e^{-2x}$, find $\frac{d^2y}{dx^2}$ at $x = 3$.
29. If , find at

a. b. c. 1 d. 0

30. If $y = f(u)$ and $u = x^2 + 1$, then find $\frac{dy}{dx}$

a. $f'(u).2x$ b. $f(u)x^2$ c. d.

ANSWERS

1. $\frac{1}{2\sqrt{x}}$	2. $-\frac{5}{2}x^{-\frac{7}{2}}$	3. $-1+6x$
4. $2x+2$	5. $\frac{2}{x} - 5\sec x \tan x$	6. $1-7x^{-2}$
7. $-(2x+1)\sin(x^2+x+1)$	8. $3\sin^2 x.\cos x$	9. $-\sin(\sin x).\cos x$
10. $\frac{1}{\sin x \cos x}$	11. $10t\,e^{5t^2}$	12. $2s.2^{s^2+9}.\log_e 2$

13. $x^2 \cos x + 2x \sin x$

14. $\frac{\cos x}{x} - \log x . \sin x$

15. $2^t \left\{ (3t^2 - 9)\log_e 2 + 6t \right\}$

16. $\frac{(x+1) - x\log(7x)}{x(x+1)^2}$

17. $\frac{\tan x - x\log x . \sec^2 x}{x\tan^2 x}$

18. $\frac{2x\sin x - (x^2+1)\cos x}{\sin^2 x}$

19. $\frac{2(\sec^2 2x - \tan 2x)}{e^{2x}}$

20. $\frac{3}{4}t$ or $\frac{3}{4}\sqrt{\frac{x-1}{2}}$

21. $2t\sec^2 t$

22. $\frac{5^t \log_e 5 + 7}{\sec t . \tan t}$

23. $(\cos x)^x \left(-x\tan x + \log(\cos x)\right)$

24. $x^{\cos x}\left(\frac{\cos x}{x} - \log x \sin x\right)$

25. $\frac{dy}{dx} = \frac{1}{(2y-1)}$ where $y = \sqrt{x + \sqrt{x + \sqrt{x + \sqrt{x + \ldots}}}}$

26. $\frac{dy}{dx} = \frac{-y^2 \tan x}{(1 - y\log(\cos x))}$ where

27. $-\operatorname{cosec}^2 x + 25e^{5x}$

28. $18e^{-6}$

29. c

30. a

4.5 APPLICATIONS OF DIFFERENTIAL CALCULUS

(a) Derivative as Rate Measures:

Let y be a function of x, then $\frac{dy}{dx}$ represents the rate of change of y with respect to x.

If $\frac{dy}{dx} > 0$ then y increases when x changes and if $\frac{dy}{dx} < 0$ then y decreases when x changes.

Some Important Points to Remember:

(i) Usually **t** , **s** , **v** and **a** are used to represent time, displacement, velocity and acceleration respectively.

Also $v = \frac{ds}{dt}$

$a = \frac{dv}{dt} = \frac{d^2 s}{dt^2} = v . \frac{dv}{ds}$

(ii) If the particle moves in the direction of s increasing, then $v = \frac{ds}{dt} > 0$ and if the particle moves in the direction of s decreasing, then $v = \frac{ds}{dt} < 0$.

(iii) If $a = 0$ then the particle is said to be moving with constant velocity and if $a < 0$ then the particle is said to have retardation.

(iv) If $\frac{dy}{dx} = 0$ then y is constant.

(v) If $y = f(x)$ be a curve then $\frac{dy}{dx}$ is said to be the slope of the curve. It is also represented by m i.e. $slope = m = \frac{dy}{dx}$.

(vi) If r is the radius, A is the area and C is the circumference of the circle then $A = \pi r^2$ & $C = 2\pi r$.

(vii) If r is the radius, S is the surface area and V is the volume of the sphere then $S = 4\pi r^2$ & $V = \frac{4}{3}\pi r^3$.

(viii) If r is the radius of base, h is the height, l is the slant length, S is the surface area and V is the volume of the cone then $S = \pi r l + \pi r^2$ & $V = \frac{1}{3}\pi r^2 h$.

(ix) If a is the length of side of a base, S is the surface area and V is the volume of the cube then $S = 6a^2$ & $V = a^3$.

<u>Examples Related to Rate Measure</u>:

Example 39. If $y = x^3 + 5x^2 - 6x + 7$ and x increases at the rate of 3 units per minute, how fast is the slope of the curve changes when $x = 2$.

Sol. Let t represents the time.

Given that $y = x^3 + 5x^2 - 6x + 7$ (1)

and $\frac{dx}{dt} = 3$ (2)

Let m be the slope of the curve.

$\therefore \quad m = \frac{dy}{dx}$

$\Rightarrow \quad m = \frac{d}{dx}\left(x^3 + 5x^2 - 6x + 7\right)$ (used (1.1)) (using (1))

$\Rightarrow \quad m = 3x^2 + 10x - 6$

Differentiating it with respect to t, we get

$$\frac{dm}{dt} = \frac{d}{dt}\left(3x^2 + 10x - 6\right)$$

$\Rightarrow \quad \frac{dm}{dt} = (6x + 10)\frac{dx}{dt}$

$\Rightarrow \quad \frac{dm}{dt} = (6x + 10).3$ (used (1.2)) (using (2))

$\Rightarrow \quad \frac{dm}{dt} = 18x + 30$ (3)

Put $x = 2$ in (3), we get

$$\left(\frac{dm}{dt}\right)_{x=2} = 18(2) + 30 = 36 + 30 = 66$$

Hence the slope of given curve increases at the rate of 66 units per minute when $x = 2$.

Example 40. A particle is moving along a straight line such that the displacement s after time t is given by $s = 2t^2 + t + 7$. Find the velocity and acceleration at time $t = 20$.

Sol. Let v be the velocity and a be the acceleration of the particle at time t.

Given that the displacement of the particle is $s = 2t^2 + t + 7$

Differentiating it with respect to t, we get

$$\frac{ds}{dt} = \frac{d}{dt}\left(2t^2 + t + 7\right)$$

$\Rightarrow \quad v = 4t + 1$ (1)

Again differentiating with respect to t, we get

$$\frac{dv}{dt} = \frac{d}{dt}(4t+1)$$

$\Rightarrow \quad a = 4$ (2)

Put $t = 20$ in (1) and (2), we get

$$[v]_{t=20} = 4(20)+1 = 81 \;\&\; [a]_{t=20} = 4$$

Hence velocity of the particle is 81 and acceleration is 4 when $t = 20$.

Example 41. Find the rate of change of the area of the circle with respect to its radius r when $r = 4\ cm.$.

Sol. Given that r be the radius of the circle.

Let A be the area of the circle.

$\therefore \quad A = \pi r^2$

$$\Rightarrow \quad \frac{dA}{dr} = \frac{d}{dr}\left(\pi r^2\right)$$

$$\Rightarrow \quad \frac{dA}{dr} = 2\pi r$$ (1)

Put $r = 4$ in (1), we get

$$\left(\frac{dA}{dr}\right)_{r=4} = 2\pi \times 4 = 8\pi$$

Hence the rate of change of area of the circle is $8\pi\ cm^2/.\sec$

Example 42. Find the rate of change of the surface area of a ball with respect to its radius r.

Sol. Given that r is the radius of the ball.

Let S be the surface area of the ball.

$\therefore \quad S = 4\pi r^2$

$$\Rightarrow \quad \frac{dS}{dr} = \frac{d}{dr}\left(4\pi r^2\right)$$

$$\Rightarrow \quad \frac{dS}{dr} = 4\pi \times 2r = 8\pi r$$

which is the required rate of change of the surface area of a ball with respect to its radius r.

Example 43. The radius of an air bubble increases at the rate of $2\ cm/\sec$. At what rate is the volume of the bubble increases when the radius is $5\ cm$?

Sol. Let r be the radius, V be the volume of the bubble and t represents time.

So, by given statement $\frac{dr}{dt} = 2\ cm/\sec$

(1)

and $\quad V = \frac{4}{3}\pi r^3$

$$\Rightarrow \quad \frac{dV}{dt} = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)$$

$$\Rightarrow \quad \frac{dV}{dt} = \frac{4}{3}\pi \times 3r^2 \times \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\Rightarrow \quad \frac{dV}{dt} = 8\pi r^2 \qquad (2)$$

Put $r = 5$ in (2), we get

$$\left(\frac{dV}{dt}\right)_{r=5} = 8\pi \times (5)^2 = 200\pi$$

Hence, volume of the bubble increases at the rate of $200\pi\, cm^3 / \sec$.

Example 44. Find the rate of change of the volume of the cone with respect to the radius of its base.

Sol. Let r be the radius of the base, h be the height and V be the volume of the cone.

$$\therefore \quad V = \frac{1}{3}\pi r^2 h$$

$$\Rightarrow \quad \frac{dV}{dr} = \frac{d}{dr}\left(\frac{1}{3}\pi r^2 h\right)$$

$$\Rightarrow \quad \frac{dV}{dr} = \frac{1}{3}\pi h \times 2r = \frac{2}{3}\pi r h.$$

Example 45. Find the rate of change of the surface area of the cone with respect to the radius of its base.

Sol. Let r be the radius of the base, l be the slant length and S be the surface area of the cone.

$$\therefore \quad S = \pi r l + \pi r^2$$

$$\Rightarrow \quad \frac{dS}{dr} = \frac{d}{dr}\left(\pi r l + \pi r^2\right)$$

$$\Rightarrow \quad \frac{dS}{dr} = \pi l + \pi \times 2r$$

$$\Rightarrow \quad \frac{dS}{dr} = \pi l + 2\pi r$$

Example 46. Sand is pouring from a pipe at the rate $10\, cc/\sec$. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-fifth of the radius of the base. How fast the height of the sand cone increases when the height is $6\, cm$?

Sol. Let r be the radius of the base, h be the height and V be the volume of the cone at the time t.

So, by given statement $h = \frac{r}{5}$ (1)

and $V = \frac{1}{3}\pi r^2 h$

$$\Rightarrow \quad V = \frac{1}{3}\pi (5h)^2 h = \frac{25}{3}\pi h^3 \qquad (used(using(15.(11))))$$

$$\Rightarrow \quad \frac{dV}{dt} = \frac{d}{dt}\left(\frac{25}{3}\pi h^3\right) = \frac{25}{3}\pi \times 3h^2 \frac{dh}{dt} = 25\pi h^2 \frac{dh}{dt} \qquad (2)$$

Also, by given statement $\frac{dV}{dt} = 10\ cc/\sec$ (3)

From (2) and (3), we get

$$25\pi h^2 \frac{dh}{dt} = 10$$

$$\Rightarrow \frac{dh}{dt} = \frac{10}{25\pi h^2} = \frac{2}{5\pi h^2}$$

When $h = 6\ cm$, $\frac{dh}{dt} = \frac{2}{5\pi(6)^2} = \frac{1}{90\pi}$

Hence, the rate of increase of height of the sand cone is $\frac{1}{90\pi}\ cm/\sec$, when $h = 6\ cm$.

Example 47. The length of edges of a cube increases at the rate of $2\ cm/\sec$. At what rate is the volume of the cube increases when the edge length is $1 cm$?

Sol. Let a be the length of edge and V be the volume of the cube at time t.

So, by given statement $\frac{da}{dt} = 2\ cm/\sec$

(1)

and $V = a^3$

$$\Rightarrow \frac{dV}{dt} = \frac{d}{dt}\left(a^3\right)$$

$$\Rightarrow \frac{dV}{dt} = 3a^2 \frac{da}{dt} \quad \text{(using (1))}$$

$$\Rightarrow \frac{dV}{dt} = 6a^2 \quad (2)$$

Put $a = 1$ in (2), we get

$$\left(\frac{dV}{dt}\right)_{a=1} = 6(1)^2 = 6$$

Hence, volume of the cube increases at the rate of $6\ cm^3/\sec$.

(b) Maxima and Minima

Maximum Value of a Function & Point of Maxima: Let $f(x)$ be a function defined on domain $D \subset R$. Let a be any point of domain D. We say that $f(x)$ has maximum value at a if $f(x) \le f(a)$ for all $x \in D$ and a is called the point of maxima.

e.g.Let	$f(x) = -x^2 + 5$	for all $x \in R$
Now	$x^2 \ge 0$	for all $x \in R$
	$-x^2 \le 0$	for all $x \in R$
	$-x^2 + 5 \le 5$	for all $x \in R$
i.e.	$f(x) \le 5$	for all $x \in R$

Hence 5 is the maximum value of $f(x)$ which is attained at $x = 0$. Therefore, $x = 0$ is the point of maxima.

Minimum Value of a Function & Point of Minima: Let $f(x)$ be a function defined on domain $D \subset R$. Let a be any point of domain D. We say that $f(x)$ has minimum value at a if $f(x) \ge f(a)$ for all $x \in D$ and a is called the point of minima.

e.g. Let $f(x) = x^2 + 8$ for all $x \in R$

Now $x^2 \geq 0$ for all $x \in R$

$x^2 + 8 \geq 8$ for all $x \in R$

i.e. $f(x) \geq 8$ for all $x \in R$

Hence 8 is the minimum value of $f(x)$ which is attained at $x = 0$. Therefore, $x = 0$ is the point of minima.

Note: We can also attain points of maxima and minima and their corresponding maximum and minimum value of a given function by differential calculus too.

Working Rule to find points of maxima or minima or inflexion by Differential Calculus:

Step No.	Working Procedure
1	Put $y = f(x)$
2	Find $\frac{dy}{dx}$
3	Put $\frac{dy}{dx} = 0$ and solve it for x. Let $x_1, x_2, \ldots, x_n$ are the values of x.
4	Find $\frac{d^2y}{dx^2}$.
5	Put the values of x in $\frac{d^2y}{dx^2}$. Suppose $x = x_i$ be any value of x. If $\frac{d^2y}{dx^2} < 0$ at $x = x_i$ then $x = x_i$ is the point of maxima and $f(x_i)$ is the maximum value of $f(x)$. If $\frac{d^2y}{dx^2} > 0$ at $x = x_i$ then $x = x_i$ is the point of minima and $f(x_i)$ is minimum value of $f(x)$. If $\frac{d^2y}{dx^2} = 0$ at $x = x_i$. Find $\frac{d^3y}{dx^3}$. If $\frac{d^3y}{dx^3} \neq 0$ at $x = x_i$ then $x = x_i$ is the point of inflexion.

Examples of Maxima and Minima:

Example 48. Find all the points of maxima and minima and the corresponding maximum and minimum values of the function $f(x) = x^3 - 12x^2 + 5$.

Sol. Let $y = f(x) = x^3 - 12x^2 + 5$

Differentiating it with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^3 - 12x^2 + 5\right)$$

Again differentiating with respect to x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(3x^2 - 24x\right)$$

$$\frac{d^2 y}{dx^2} = 6x - 24$$

Put $\frac{dy}{dx} = 0$, we get

$$3x^2 - 24x = 0$$

$$3x(x-8)=0$$

Either $x=0$ *or* $x-8=0$

i.e. *Either* $x=0$ *or* $x=8$

When $x=0$:

$$\left(\frac{d^2 y}{dx^2}\right)_{x=0} = (6x-24)_{x=0} = -24 < 0$$

which shows that $x=0$ is a point of maxima.

So maximum value of $f(x) = x^3 - 12x^2 + 5$ is

$$(y)_{x=0} = (x^3 - 12x^2 + 5)_{x=0} = 5$$

When $x=8$:

$$\left(\frac{d^2 y}{dx^2}\right)_{x=8} = (6x-24)_{x=8} = 6(8) - 24 = 24 > 0$$

which shows that $x=8$ is a point of minima.

So minimum value of $f(x) = x^3 - 12x^2 + 5$ is

$$(y)_{x=8} = (x^3 - 12x^2 + 5)_{x=8} = 8^3 - 12(8)^2 + 5$$

$$= 512 - 768 + 5 = -251$$

Example 49. Find all the points of maxima and minima and the corresponding maximum and minimum values of the function $f(x) = \sin x + \cos x$ *where* $0 \le x \le \frac{\pi}{2}$.

Sol. Let $y = f(x) = \sin x + \cos x$

Differentiating it with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(\sin x + \cos x)$$

$$\frac{dy}{dx} = \cos x - \sin x$$

Again differentiating with respect to x, we get

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}(\cos x - \sin x)$$

$$\frac{d^2 y}{dx^2} = -\sin x - \cos x$$

Put $\frac{dy}{dx} = 0$, we get

$$\cos x - \sin x = 0$$

$$\cos x = \sin x$$

$$\frac{\cos x}{\sin x} = 1$$

$$\cot x = 1$$

$\cot x = \cot \frac{\pi}{4}$ as $0 \le x \le \frac{\pi}{2}$

$x = \frac{\pi}{4}$

When $x = \frac{\pi}{4}$:

$$\left(\frac{d^2 y}{d x^2}\right)_{x=\frac{\pi}{4}} = (-\sin x - \cos x)_{x=\frac{\pi}{4}} = -\sin\frac{\pi}{4} - \cos\frac{\pi}{4}$$

$$= -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\frac{2}{\sqrt{2}} < 0$$

which shows that $x = \frac{\pi}{4}$ is a point of maxima.

So maximum value of $f(x) = \sin x + \cos x$ is

$$(y)_{x=\pi/4} = (\sin x + \cos x)_{x=\pi/4}$$

$$= \sin\frac{\pi}{4} + \cos\frac{\pi}{4}$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}}$$

Example 50. Find two positive numbers x & y such that $x \cdot y = 16$ and the sum $x + y$ is minimum. Also find the minimum value of sum.

Sol. Given that $x \cdot y = 16$ $\Rightarrow y = \frac{16}{x}$

Let $S = x + y$ $\Rightarrow S = x + \frac{16}{x}$

Differentiating it with respect to x, we get

$$\frac{dS}{dx} = \frac{d}{dx}\left(x + \frac{16}{x}\right)$$

$$\frac{dS}{dx} = 1 + 16\left(-\frac{1}{x^2}\right) = 1 - \frac{16}{x^2}$$

Again differentiating with respect to x, we get

$$\frac{d^2 S}{dx^2} = \frac{d}{dx}\left(1 - \frac{16}{x^2}\right)$$

$$\frac{d^2 S}{dx^2} = 0 - 16\left(-\frac{2}{x^3}\right) = \frac{32}{x^3}$$

Put $\frac{dS}{dx} = 0$, we get

$$1 - \frac{16}{x^2} = 0$$

$$\frac{x^2 - 16}{x^2} = 0$$

$$x^2 - 16 = 0$$

Either $x = 4$ *or* $x = -4$

$x = -4$ is rejected as x is positive.

When $x = 4$:

$$\left(\frac{d^2S}{dx^2}\right)_{x=4} = \left(\frac{32}{x^3}\right)_{x=4} = \frac{32}{4^3} = \frac{32}{64} = \frac{1}{2} > 0$$

which shows that $x = 4$ is a point of minima.

Now at $x = 4$, value of y is :

$$(y)_{x=4} = \left(\frac{16}{x}\right)_{x=4} = \frac{16}{4} = 4$$

Also minimum value of sum $S = x + y$ is

$$(S)_{x=4,\, y=4} = (x+y)_{x=4,\, y=4} = 4+4 = 8$$

Example 51. Find the dimensions of the rectangle of given area 169 sq**.** cm. whose perimeter is least. Also find its perimeter.

Sol. Let the sides of the rectangle be and , A be the area and P be the perimeter.

$\therefore \quad A = xy = 169\ sq.c.m. \qquad \Rightarrow y = \frac{169}{x}$

And $\quad P = 2(x+y) \qquad \Rightarrow P = 2\left(x + \frac{169}{x}\right) = 2x + \frac{338}{x}$

Differentiating it with respect to x, we get

$$\frac{dP}{dx} = \frac{d}{dx}\left(2x + \frac{338}{x}\right)$$

$$\frac{dP}{dx} = 2 + 338\left(-\frac{1}{x^2}\right) = 2 - \frac{338}{x^2}$$

Again, differentiating with respect to x, we get

$$\frac{d^2P}{dx^2} = \frac{d}{dx}\left(2 - \frac{338}{x^2}\right)$$

$$\frac{d^2P}{dx^2} = 0 - 338\left(-\frac{2}{x^3}\right) = \frac{676}{x^3}$$

Put $\frac{dP}{dx} = 0$, we get

$$2 - \frac{338}{x^2} = 0$$

$$\frac{2x^2 - 338}{x^2} = 0$$

$$2x^2 - 338 = 0$$

$$x^2 = 169$$

Either $x = 13$ *or* $x = -13$

$x = -13$ is rejected as x can"t be negative.

When $x = 13$:

$$\left(\frac{d^2P}{dx^2}\right)_{x=13} = \left(\frac{676}{x^3}\right)_{x=13} = \frac{676}{(13)^3} = \frac{4}{13} > 0$$

which shows that $x = 13$ is a point of minima.

Therefore, Perimeter is least at $x = 13$.

Now at $x = 13$, value of y is :

$$(y)_{x=13} = \left(\frac{169}{x}\right)_{x=13} = \frac{169}{13} = 13$$

Also least value of perimeter $P = 2(x+y)$ is

$$(P)_{x=13,\, y=13} = (2x+2y)_{x=13,\, y=13} = 26 + 26 = 52\ c\text{cm..}$$

Example 52. Show that among all the rectangles of a given perimeter, the square has the maximum area.

Sol. Let the sides of the rectangle are x and y, A be the area and P be the given perimeter.

$\therefore \quad P = 2(x+y) \quad \Rightarrow P = 2x + 2y \quad \Rightarrow y = \frac{P-2x}{2} = \frac{P}{2} - x$

and

$$A = xy = \frac{Px}{2} - x^2$$

Differentiating it with respect to x, we get

$$\frac{dA}{dx} = \frac{d}{dx}\left(\frac{Px}{2} - x^2\right)$$

$$\frac{dA}{dx} = \frac{P}{2} - 2x$$

Again differentiating with respect to x, we get

$$\frac{d^2A}{dx^2} = \frac{d}{dx}\left(\frac{P}{2} - 2x\right)$$

$$\frac{d^2A}{dx^2} = 0 - 2 = -2$$

Put $\frac{dA}{dx} = 0$, we get

$$\frac{P}{2} - 2x = 0$$

$$x = \frac{P}{4}$$

When $x = \frac{P}{4}$:

$$\left(\frac{d^2A}{dx^2}\right)_{x=\frac{P}{4}} = -2 < 0$$

which shows that $x = \frac{P}{4}$ is a point of maxima.

Therefore, Area is maximum at $x = \frac{P}{4}$.

Now at $x = \frac{P}{4}$, value of y is :

$$(y)_{x=\frac{P}{4}} = \left(\frac{P}{2} - x\right)_{x=\frac{P}{4}} = \frac{P}{2} - \frac{P}{4} = \frac{P}{4}$$

$\Rightarrow \quad x = y = \frac{P}{4}$ gives the maximum area.

Hence, among all the rectangles of a given perimeter, the square has the maximum area.

Example 53. Find all the points of maxima and minima and the corresponding maximum and minimum values of the function $f(x) = x^3 + 1$.

Sol. Let $y = f(x) = x^3 + 1$

Differentiating it with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^3 + 1\right)$$

Again, differentiating with respect to x, we get

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(3x^2\right)$$

$$\frac{d^2 y}{dx^2} = 6x$$

Put $\frac{dy}{dx} = 0$, we get

$$3x^2 = 0$$

$$\Rightarrow \quad x = 0$$

When $x = 0$:

$$\left(\frac{d^2 y}{dx^2}\right)_{x=0} = (6x)_{x=0} = 6 \times 0 = 0$$

To check maxima or minima, we need to find third order derivative of y with respect to x.

So, $\frac{d^3 y}{dx^3} = \frac{d}{dx}(6x)$

$$\Rightarrow \quad \frac{d^3 y}{dx^3} = 6 \neq 0$$

which shows that $x = 0$ is neither a point of maxima nor a point of minima, hence the given function has neither maximum value nor minimum value.

EXERCISE-III

1. If $y = 5 - 3x^2 + 2x^3$ and x decreases at the rate of 6 units per seconds, how fast is the slope of the curve changes when $x = 7$.
2. If a particle is moving in a straight line such that the displacement s after time t is given by $s = \frac{1}{2} v t$, where v be the velocity of the particle. Prove that the acceleration a of the particle is constant.
3. The radius of the circle increases at the rate $0.4\ cm/\sec$. What is the increase of its circumference.
4. Find the rate of change per second of the volume of a ball with respect to its radius r when $r = 6\ cm$.
5. Find the rate of change per minute of the surface area of a ball with respect to its radius r when $r = 9\ m$.

6. Find the rate of change of the volume of the cone with respect to its height.

7. Find all the points of maxima and minima and the corresponding maximum and minimum values of the function $f(x) = 6x^3 - 27x^2 + 36x + 6$.

8. Find all the points of maxima and minima and the corresponding maximum and minimum values of the function $f(x) = -2x^3 + 6x^2 + 18x - 1$.

9. Find all the points of maxima and minima and the corresponding maximum and minimum values of the function $f(x) = \frac{\log x}{x}$ $if\ 0 < x < \infty$.

10. The maximum value of the function $y = -x^3 + 1$ over the interval [-2,1]

    b. 9 b. 0 c. 2 d. 10

11. Find the length (l) and breadth (b) of a rectangle with perimeter 12 such that it has maximum area.

    a. 3,3 b. 2,4 c. 3,4 d. 5,2

ANSWERS

1. -468 3. $0.8\pi\, cm/\sec$ 4. $144\pi\, cm^3/\sec$

5. $72\pi m^2/\min$ 6. $\frac{1}{3}\pi r^2$

7. $x = 1$ is a point of maxima and maximum value of function is 21, $x = 2$ is a point of minima and minimum value of function is 18.

8. $x = -1$ is a point of minima minimum value of function is -11, $x = 3$ is a point of maxima and maximum value of function is 53.

9. $x = e$ is a point of maxima and maximum value of function is $\frac{1}{e}$.

10. b 11. a

UNIT – 5

INTEGRAL CALCULUS

Learning Objectives

- To learn the concept of integration and its geometrical meaning.
- To learn various formulae to evaluate the integrals.
- To learn about definite integral and its application to calculate the area under the curves.

5.1 INTEGRATION – Reverse operation of differentiation

Integration is the reverse process of differentiation. In the previous chapter, we have studied differentiation as the study of small change in one variable with respect to small change in other. In the same manner, integration is the study of a function as a whole when small changes are given. For example

if dA shows small change in area, then $\int dA$ is the area as a whole.

If f(x) and g(x) are two functions such that $\frac{dg(x)}{dx} = f(x)$, then $\int f(x)dx = g(x) + c$ i.e. g(x) is integral of f(x) with respect to x and c is constant of integration.

The function to be integrated is called integrand and put in between the sign $\int dx$.

Main Rule of Integration: We can integrate a function if it is in single form (variables/functions are not in product or quotient form) otherwise we will have to use various different methods to convert it in single form and then integrate.

5.2 SIMPLE STANDARD INTEGRAL

(a)Integral of algebraic functions:

$\int 0\, dx =$ const**an**tt

$\int 1dx = x + c$

$\int x^n dx = \frac{x^{n+1}}{n+1} + c, n \neq -1$n, is any real number $(n \neq -1)$

$\int \frac{1}{x} dx = \log x + c$

Some Results of Integration

I. $\frac{d}{dx}\int (f(x)dx) = f(x)$

II. $\int kf(x)\,dx = k\int f(x)dx$ where k is any constant.

III. $\int (f(x) \pm g(x))dx = \int f(x)\,dx \pm \int g(x)\,dx$

IV. $\int kdx = kx + c$ where k is any constant and c constant of integration

Example 1. Evaluate (i) $\int x^3 dx$ (ii) $\int \sqrt{x}\,dx$ (iii) $\int \frac{1}{\sqrt{x}}dx$ (iv) $\int 2dx$

(v) $\int (2x^2 + 3x + 5)dx$ (vi) $\int (x^3 - 5x + \frac{1}{x})dx$

Sol : (i) $\int x^3 dx = \frac{x^4}{4} + c$ (iv) $\int 2dx = 2x + c$

(ii) $\int \sqrt{x}\,dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$ (v) $\int (2x^2 + 3x + 5)dx = 2\frac{x^3}{3} + 3\frac{x^2}{2} + 5x + c$

(iv) $\int \frac{1}{\sqrt{x}}dx = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$ (vi) $\int (x^3 - 5x + \frac{1}{x})dx = \frac{x^4}{4} - 5\frac{x^2}{2} + \log x + c$

Example 2. Evaluate (i) $\int (x+1)x^2\,dx$ (ii) $\int \frac{x^2+1}{x}dx$ (iii) $\int \frac{x^3 - x + 3}{x}dx$

(iv) $\int (\sqrt{x} + \frac{1}{\sqrt{x}})dx$

Sol : (i) $\int (x+1)x^2\,dx = \int (x^3 + x^2)dx = \frac{x^4}{4} + \frac{x^3}{3} + c$

(ii) $\int \frac{x^2+1}{x}dx = \int \left(x + \frac{1}{x}\right)dx = \frac{x^2}{2} + \log x + c$

(iii) $\int \frac{x^3 - x + 3}{x}dx = \int (x^2 - 1 + \frac{3}{x})dx = \frac{x^3}{3} - x + \log x + c$

(iv) $\int \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)dx = \frac{x^{3/2}}{3/2} - \frac{x^{1/2}}{1/2} + c$

EXERCISE-I

Integrate the following functions with respect to x

1. Evaluate

   i. $\int x^4 dx$ ii. $\int x^{\frac{5}{4}} dx$ iii. $\int \frac{1}{x^5} dx$ iv. $\int \frac{1}{x^{3/4}} dx$

2. $\int (3x\sqrt{x} + 4\sqrt{x} + 5)dx$

3. $\int \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 dx$

4. $\int \frac{x^3 + 5x^2 + 4x + 1}{x^2} dx$

5. $\int \left(x + \frac{1}{x}\right)^3 dx$

6. $\int \frac{(1+x)^2}{\sqrt{x}} dx$

7. $\int \frac{(x+1)(x-2)}{\sqrt{x}} dx$

ANSWERS

1. (i) $\frac{x^5}{5} + c$ (ii) $\frac{4}{9} x^{\frac{9}{4}} + c$ (iii) $\frac{x^{-4}}{-4} + c$ (iv) $4x^{\frac{1}{4}} + c$

2. $\frac{6}{5} x^{\frac{5}{2}} + \frac{8}{3} x^{\frac{3}{2}} + 5x + c$

3. $\frac{x^2}{2} + \log x + 2x + c$

4. $\frac{x^2}{2} + 5x + 4 \log x - \frac{1}{x} + c$

5. $\frac{x^4}{4} - \frac{1}{2x^2} + 3\frac{x^2}{2} + 3 \log x + c$

6. $2x^{\frac{1}{2}} + \frac{2}{5} x^{\frac{5}{2}} + \frac{4}{3} x^{\frac{3}{2}} + c$

7. $\frac{2}{5} x^{\frac{5}{2}} - \frac{2}{3} x^{\frac{3}{2}} - 4x^{\frac{1}{2}} + c$

(b) Integrals of the type $(ax + b)^n$

When $\int n \neq -1,$ $\quad (ax+b)^n \, dx = \dfrac{(ax+b)^{n+1}}{(n+1)a} + c, \; n \neq -1$, n is any real number

When $\quad , \int \dfrac{1}{ax+b} dx = \dfrac{\log(ax+b)}{a} + c$

Example 3. Evaluate (i) $\int (1-3x)^5 \, x \, dx$ $\qquad$ (ii) $\int \sqrt{2x+3} \, dx$

(iii) $\int \dfrac{1}{3+2x} dx$ $\qquad$ (iv) $\int \dfrac{1}{(5+3x)} dx$

Sol : (i) $\int (1-3x)^5 \, dx = \dfrac{(1-3x)^6}{6(-3)} + c = \dfrac{(1-3x)^6}{-18} + c$

(ii) $\int 2x+3 \, dx = \dfrac{(2x+3)^{3/2}}{\frac{3}{2}2} + c = \dfrac{(2x+3)^{3/2}}{3} + c$

(iii)

(iv) $\int \dfrac{1}{(5+3x)^3} dx = \int (5+3x)^{-3} dx = \dfrac{(5+3x)^{-2}}{-2 \times 3} + c = \dfrac{(5+3x)^{-2}}{-6} + c$

EXERCISE-II

Integrate the following functions with respect to x

1. $\int \dfrac{x^3 - x + 3}{x+1} dx$

2. $\int \sqrt{3-2x} \, dx$

3. $\int (5+3x)^{-7}\,dx$

4. $\int (5x+3)^{4}\,dx$

5. $\int \frac{1}{\sqrt{1-4x}}\,dx$

6. $\int \frac{1}{\sqrt{2x+5}+\sqrt{2x-5}}\,dx$

7. $\int \frac{x^2+5x+2}{x+2}\,dx$

8. $\int \left(\frac{1}{2-3x}+\frac{1}{\sqrt{3x-2}}\right)dx$

9. $\int \sqrt{3+2x}+(3x-5)^{4}\,dx$

10. $\int \frac{x-1}{\sqrt{x+4}}\,dx$

ANSWERS

1. $\frac{x^3}{3}-\frac{x^2}{2}+3\log(x+1)+c$ 2. $\frac{(3-2x)^{3/2}}{-3}+c$ 3. $\frac{(5+3x)^{-8}}{-24}+c$

4. $\frac{(5x+3)^5}{25}+c$ 5. $\frac{(1-4x)^{1/2}}{-2}+c$ 6. $\frac{1}{30}[(2x+5)^{3/2}-(2x-5)^{3/2}]+c$

7. $\frac{x^2}{2}+3x-4\log(x+2)+c$ 8. $\frac{\log(2-3x)}{-3}+\frac{2(3x-2)^{1/2}}{3}+c$

9. $\frac{(3+2x)^{3/2}}{3}+\frac{(3x-5)^5}{15}+c$ 10. $\frac{2(x+4)^{3/2}}{3}-10(x+4)^{1/2}+c$

(c) More formulae of integrals of the following type:

1. $\int \frac{1}{x^2+a^2}\,dx=\frac{1}{a}\tan^{-1}\frac{x}{a}+c$

2. $\int \frac{-1}{x^2+a^2}\,dx=\frac{1}{a}\cot^{-1}\frac{x}{a}+c$

3. $\int \frac{1}{\sqrt{x^2+a^2}}dx = \log|x+\sqrt{x^2+a^2}|+c$

4. $\int \frac{1}{\sqrt{x^2-a^2}}dx = \log|x+\sqrt{x^2-a^2}|+c$

5. $\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a}+c$

6. $\int \frac{-1}{\sqrt{a^2-x^2}}dx = \cos^{-1}\frac{x}{a}+c$

7. $\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\log\frac{x-a}{x+a}+c$

8. $\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\log\frac{a+x}{a-x}+c$

9. $\int \frac{1}{x\sqrt{x^2-a^2}}dx = \frac{1}{a}\sec^{-1}\frac{x}{a}+c$

10. $\int \frac{-1}{x\sqrt{x^2-a^2}}dx = \frac{1}{a}\operatorname{cosec}^{-1}\frac{x}{a}+c$

Example 4. Evaluate (i) $\int \frac{dx}{x^2+9}$ (ii) $\int \frac{dx}{2x^2+9}$ (iii) $\int \frac{dx}{\sqrt{1-3x^2}}$

(iv) $\int \frac{dx}{\sqrt{4x^2+16}}$ (v) $\int \frac{dx}{x^2-25}$ (vi) $\int \frac{dx}{9x^2-16}$

Sol : (i) $\int \frac{dx}{x^2+9} = \int \frac{dx}{x^2+3^2}$

Using formula $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}+c$

Here a = 3

So $\int \frac{dx}{x^2+3^2} = \frac{1}{3}\tan^{-1}\frac{x}{3}+c$

(ii) $\int \frac{dx}{2x^2+9} = \frac{1}{2}\int \frac{dx}{\frac{2x^2+9}{2}} = \frac{1}{2}\int \frac{dx}{x^2+\frac{9}{2}} = \frac{1}{2}\int \frac{dx}{x^2+\left(\frac{3}{\sqrt{2}}\right)^2}$

Using formula $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

$a = \frac{3}{\sqrt{2}}$

$\therefore \frac{1}{2}\int \frac{dx}{x^2+\left(\frac{3}{\sqrt{2}}\right)^2} = \frac{1}{2}\left[\frac{1}{3/\sqrt{2}}\tan^{-1}\frac{x}{3/\sqrt{2}}\right] + c$

$= \frac{1}{3\sqrt{2}}\tan^{-1}\frac{\sqrt{2}x}{3} + c$

(iii) $\int \frac{dx}{\sqrt{1-9x^2}} = \int \frac{dx}{\sqrt{9\left(\frac{1-9x^2}{9}\right)}} = \int \frac{dx}{3\sqrt{\frac{1}{9}-x^2}}$

$= \frac{1}{3}\int \frac{dx}{\sqrt{\left(\frac{1}{3}\right)^2 - x^2}}$ Here $a = \frac{1}{3}$

Using formula $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c$

$\frac{1}{3}\int \frac{dx}{\sqrt{\left(\frac{1}{3}\right)^2 - x^2}} = \frac{1}{3}\left[\sin^{-1}\frac{x}{1/3}\right] + c = \frac{1}{3}\left[\sin^{-1}3x\right] + c$

(iv) $\int \frac{dx}{\sqrt{4x^2+16}} = \int \frac{dx}{\sqrt{4\left(\frac{4x^2+16}{4}\right)}}$

$= \int \frac{dx}{2\sqrt{x^2+4}} = \frac{1}{2}\int \frac{dx}{\sqrt{x^2+2^2}}$

Here a = 2. Using formula $\int \frac{dx}{\sqrt{x^2+a^2}} = \log|x+\sqrt{x^2+a^2}| + c$

$\frac{1}{2}\int \frac{dx}{\sqrt{x^2+2^2}} = \frac{1}{2}\left[\log|x+\sqrt{x^2+2^2}|\right] + c$

(v) $\int \frac{dx}{\sqrt{x^2-25}} = \int \frac{dx}{\sqrt{x^2-5^2}}$

Using formula $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log | x + \sqrt{x^2 - a^2} | + c$

$$\therefore \quad \frac{dx}{\sqrt{x^2 - 5^2}} = \log | x + \sqrt{x^2 - 5^2} | + c$$

(vi) $$\int \frac{dx}{9x^2 - 16} = \int \frac{dx}{\frac{9x^2 - 16}{9} \times 9} = \frac{1}{9}\int \frac{dx}{x^2 - \frac{16}{9}}$$

$$\frac{1}{9}\int \frac{dx}{x^2 - \left(\frac{4}{3}\right)^2} \text{. Here } a = \frac{4}{3}$$

Using formula $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \frac{x - a}{x + a} + c$

$$\therefore \quad \frac{1}{9}\int \frac{dx}{x^2 - \left(\frac{4}{3}\right)^2} = \frac{1}{9}\left[\frac{1}{2 \times \frac{4}{3}} \log \frac{x - \frac{4}{3}}{x + \frac{4}{3}}\right] + c$$

$$= \frac{1}{9}\left[\frac{3}{8} \log \frac{3x - 4}{3x + 4}\right] + c$$

$$= \frac{1}{24} \log \frac{3x - 4}{3x + 4} + c$$

EXERCISE-III

1. Integrate the following integral with respect to x.

(i) $\int \frac{dx}{\sqrt{9 - x^2}}$ (ii) $\int \frac{dx}{x\sqrt{x^2 - 25}}$ (iii) $\int \frac{dx}{\sqrt{16x^2 - 36}}$

(iv) $\int \frac{dx}{25x^2 - 16}$ (v) $\int \frac{dx}{x^2 + 49}$ (vi) $\int \frac{dx}{4 + 9x^2}$

2. Evaluate $\int \frac{x^2}{1 + x^2} dx$

3. Integrate $\frac{x^6+2}{x^2+1}$ with respect to x.

4. Integrate $\left(1+\frac{1}{1+x}-\frac{2}{\sqrt{1-x^2}}+\frac{5}{x\sqrt{x^2-1}}\right)$ with respect to x.

ANSWERS

1. (i) $\sin^{-1}\frac{x}{3}+c$ (ii) $\frac{1}{5}\sec^{-1}\frac{x}{5}+c$ (iii) $\frac{1}{4}\log\left|x+\sqrt{x^2-\left(\frac{3}{2}\right)^2}\right|+c$

(iv) $\frac{1}{40}\log\frac{5x-4}{5x+4}+c$ (v) $\frac{1}{7}\tan^{-1}\frac{x}{7}+c$ (vi) $\frac{1}{6}\tan^{-1}\frac{3x}{2}+c$

2. $x-\tan^{-1}x+c$ 3. $\frac{x^5}{5}-\frac{x^3}{3}+x+\tan^{-1}x+c$

4. $x+\tan^{-1}x-2\sin^{-1}x+5\sec^{-1}x+c$

(c) Integrals of the Exponential functions

An exponential function is of the form $(\text{constant})^{\text{variable}}$, i.e. e^x, a^x, b^{3x} etc.

(i) $\int a^{mx}\,dx=\frac{a^{mx}}{(\log a)m}+c$

(ii) $\int e^{mx}\,dx=\frac{e^{mx}}{m}+c$

Example 5. Evaluate (i) $\int 3^x\,dx$ (ii) $\int \frac{a^x}{b^x}\,dx$

(iii) $\int \frac{a^x+b^x}{a^x b^x}\,dx$ (iv) $\int \frac{2^{3x}b^x}{e^x}\,dx$

Sol : (i) $\int 3^x\,dx=\frac{3^x}{(\log 3).1}+c$

(ii) $\int \frac{a^x}{b^x} dx = \int \left(\frac{a}{b}\right)^x dx = \frac{(a/b)^x}{\log(a/b)} + c$

(iii) $\int \frac{a^x + b^x}{a^x b^x} dx = \int \left(\frac{a^x}{a^x b^x} + \frac{b^x}{a^x b^x}\right) dx$

$= \int \left(\frac{1}{b^x} + \frac{1}{a^x}\right) dx$

$= \int (b^{-x} + a^{-x}) dx$

$= \frac{b^{-x}}{(\log b)(-1)} + \frac{a^{-x}}{(\log a)(-1)} + c$

(iv) $\int \frac{2^{3x} b^x}{e^x} dx = \int \left(\frac{2^3 b}{e}\right)^x dx = \int \left(\frac{8b}{e}\right)^x dx$

$= \frac{\left(\frac{8b}{e}\right)^x}{\log\left(\frac{8b}{e}\right).1} + c$

EXERCISE -IV

1. Evaluate $\int (a^{2x} + b^{2x}) dx$.

2. Integrate $\frac{(a^x + b^x)^2}{a^x b^x}$ with respect to x.

3. Integrate $\left(\frac{1}{b^x} - \frac{1}{a^x}\right)$ with respect to x.

4. Evaluate $\int (x^a + e^x + e^a) dx$.

5. Integrate $\frac{2^x + 3^x}{5^x}$ with respect to x.

ANSWERS

1. $\dfrac{a^{2x}}{(\log a)2} + \dfrac{b^{2x}}{(\log b)2} + c$

2. $\dfrac{\left(\frac{a}{b}\right)^x}{\log \frac{a}{b}} + \dfrac{\left(\frac{b}{a}\right)^x}{\log \frac{b}{a}} + 2x + c$

3. $-\dfrac{b^{-x}}{(\log b)} + \dfrac{a^{-x}}{(\log a)} + c$

4. $\dfrac{x^{a+1}}{a+1} + \dfrac{e^x}{1} + e^a x + c$

5. $\dfrac{\left(\frac{2}{5}\right)^x}{\log \frac{2}{5}} + \dfrac{\left(\frac{3}{5}\right)^x}{\log \frac{3}{5}} + c$

(d) Integral of Trigonometric Functions

1. $\int \sin x \, dx = -\cos x + c$
2. $\int \cos x \, dx = \sin x + c$
3. $\int \tan x \, dx = \log \sec x + c$ or $-\log \cos x + c$
4. $\int \operatorname{cosec} x \, dx = \log |\operatorname{cosec} x - \cot x| + c$
5. $\int \sec x \, dx = \log |\sec x + \tan x| + c$
6. $\int \cot x \, dx = \log \sin x + c$ or $-\log \operatorname{cosec} x + c$
7. $\int \sec^2 x \, dx = \tan x + c$
8. $\int \operatorname{cosec}^2 x \, dx = -\cot x + c$
9. $\int \sec x \tan x \, dx = \sec x + c$
10. $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + c$

Example 6. Evaluate : (i) $\int \tan^2 x \, dx$ (ii) $\int \cot^2 x \, dx$

(iii) $\int \sin 3x \cos 2x \, dx$ (iv) $\int \dfrac{2 + 3\cos x}{\sin^2 x} dx$

(v) $\int \dfrac{1}{1 + \sin x} dx$ (vi) $\int \dfrac{dx}{1 - \cos x}$

Sol : (i) $\int \tan^2 x\,dx = \int (\sec^2 x - 1)dx = \int \sec^2 x\,dx - \int dx$

$= \tan x - x + c$

(ii) $\int \cot^2 x\,dx = \int (\mathrm{cosec}^2 x - 1)dx$

$= \int \mathrm{cosec}^2 x\,dx - \int dx = -\cot x - x + c$

(iii) $\int \sin 3x \cos 2x\,dx = \frac{1}{2}\int 2\sin 3x \cos 2x\,dx$

Using formula 2 sin A cos B = sin (A + B) + sin (A – B)

$= \frac{1}{2}\int [\sin(3x + 2x) + \sin(3x - 2x)]dx$

$= \frac{1}{2}\int (\sin 5x + \sin x)dx$

$= \frac{1}{2}\left[-\frac{\cos 5x}{5} - \frac{\sin x}{1}\right] + c$

(iv) $\int \frac{2 + 3\cos x}{\sin^2 x}\,dx$

$= \int \frac{2}{\sin^2 x}\,dx + \int \frac{3\cos x}{\sin^2 x}\,dx$

$= 2\int \mathrm{cosec}^2 x\,dx + 3\int \cot x\,\mathrm{cosec}\,x\,dx$

$= -2\cot x - 3\,\mathrm{cosec}\,x + c$

(v) $\int \frac{dx}{1 + \sin x}$

$= \int \frac{dx}{1 + \sin} \times \frac{1 - \sin x}{1 - \sin x}$

$= \int \frac{1 - \sin x}{1 - \sin^2 x}\,dx$ as $(a - b)(a + b) = a^2 - b^2$

$= \int \frac{1 - \sin x}{\cos^2 x}\,dx$

$= \int \frac{1}{\cos^2 x}\,dx - \int \frac{\sin x}{\cos^2 x}\,dx$

$= \int \sec^2 x dx - \int \tan x \sec x \, dx$

$= \tan x - \sec x + c$

(vi) $\int \frac{dx}{1-\cos x}$

$= \int \frac{1}{1-\cos x} \times \frac{1+\cos x}{1+\cos x} dx$

$= \int \frac{1+\cos x}{1-\cos^2 x} dx$

$= \int \frac{1+\cos x}{\sin^2 x} dx$

$= \int \frac{1}{\sin^2 x} dx + \int \frac{\cos x}{\sin^2 x} dx$

$= \int \operatorname{cosec}^2 x \, dx + \int \cot x \operatorname{cosec} x \, dx$

$= -\cot x - \operatorname{cosec} x + c$

EXERCISE–V

1. Find the value of the integral is …..............

a. c.

b. d.

2. The value of is ....

a. b. c. d.

3. Evaluate $\int (3\sin x - 4\cos x + \frac{5}{\cos^2 x} - \frac{6}{\sin^2 x} + \sec x) dx$.

4. Evaluate $\int (\sec^2 x + \operatorname{cosec}^2 x) dx$.

5. Evaluate $\int \frac{5\cos^2 x + 6\sin^2 x}{2\sin^2 x \cos^2 x} dx$.

6. Evaluate $\int (\tan x + \cot x)^2 dx$.

7. Evaluate $\int \frac{dx}{1 + \cos x}$.

8. Evaluate $\int \frac{dx}{1 - \sin x}$.

9. Evaluate $\int 2 \cos 3x \cos x \, dx$.

10. Evaluate $\int \sin 5x \sin 2x \, dx$.

11. Evaluate $\int \frac{\sin x}{1+\sin x} dx$.

12. Evaluate $\int \frac{\sec x}{\sec x + \tan x} dx$.

ANSWERS

1. (a) 2. (a)

3. $-3 \cos x - 4 \sin x + 5 \tan x + 6 \cot x + \log |\sec x + \tan x| + c$

4. $\tan x - \cot x + c$

5. $-\frac{5}{2} \cot x + 3 \tan x + c$

6. $\tan x - \cot x + c$

7. $-\cot x + \operatorname{cosec} x + c$

8. $\tan x + \sec x + c$

9. $\frac{\sin 4x}{4} + \frac{\sin 2x}{2} + c$

10. $\frac{1}{2}\left[-\frac{\sin 7x}{7} + \frac{\sin 3x}{3}\right] + c$

11. $x - \tan x + \sec x + c$

12. $\tan x - \sec x + c$

There are various methods to convert the product/quotient of functions into a single function like Multiple/divide/splitting/Use of formulae. If a given function can"t be integrated by the methods explained till now :-

(1) Integration by parts
(2) Substitution method
(3) Method of partial fraction

Integration by Parts Method

If f(x) and g(x) are two functions of x then

$$\int f(x)\, g(x)dx = f(x)\int g(x)\, dx - \int \left\{\frac{d}{dx} f(x) \int g(x)dx\right\}dx .$$

Here in above formula f(x) is chosen as first function and g(x) is chosen as second function.

Note : Selection of I function and II function in accordance with function which comes first in the word “ILATE”.

where I = Inverse functions

L = Logarithmic functions

A = Algebraic functions

T = Trigonometric functions

E = Exponential functions

Example 7. Evaluate (i) $\int x \sin x\, dx$ (ii) $\int x \tan^2 x\, dx$ (iii) $\int \log x\, dx$

Sol : (i) $\int x \sin x\, dx = x\int \sin x\, dx - \int \left\{\frac{d}{dx} x \int \sin x dx\right\}dx$

$$= x(-\cos x) - \int 1\{-\cos x\}dx$$

$$= -x \cos x + \int \cos x\, dx$$

$$= -x \cos x + \sin x + c$$

(ii) $\int x \tan^2 x\, dx = \int x(\sec^2 x - 1)dx$

$$= \int x \sec^2 x\, dx - \int x\, dx$$

$$= x\int \sec^2 x\, dx - \int \left\{\frac{d}{dx} x \int \sec^2 x\, dx\right\}dx - \frac{x^2}{2}$$

$$= x \tan x - \int \{1 \tan x\}dx - \frac{x^2}{2}$$

$$= x \tan x - \int \tan x\, dx - \frac{x^2}{2}$$

$$= x \tan x - \log \sec x - \frac{x^2}{2} + c$$

(iii) $\int \log x \, dx = \int (1 \log x) dx$

$$= \int x^0 \log x \, dx$$

$$= \log x \int x^0 dx - \int \left\{ \frac{d}{dx} \log x \int x^0 \, dx \right\} dx$$

$$= (\log x)x - \int \left(\frac{1}{x} x \right) dx$$

$$= (\log x)x - \int 1 dx$$

$$= (\log x)x - x + c$$

Example 8. Evaluate (i) $\int x^2 \cos x \, dx$ (ii) $\int x^2 e^x \, dx$.

Sol : (i) $\int x^2 \cos x \, dx = x^2 \int \cos x \, dx - \int \left\{ \frac{d}{dx} x^2 \int \cos x \, dx \right\} dx$

$$= x^2 \sin x - \int 2\{x \sin x\} dx$$

$$= x^2 \sin x - 2 \int x \sin x \, dx$$

Again by parts method

$$= x^2 \sin x - 2 \left[x \int \sin x dx - \int \left\{ \frac{d}{dx} x \int \sin x \, dx \right\} dx \right]$$

$$= x^2 \sin x - 2 \left[x(-\cos x) - \int \{1 \times -\cos x\} dx \right]$$

$$= x^2 \sin x - 2 \left[-x \cos x + \int \cos x \, dx \right]$$

$$= x^2 \sin x + 2x \cos x - 2 \sin x + c$$

(ii) $\int x^2 e^x dx = x^2 \int e^x dx - \int \left\{ \frac{d}{dx} x^2 \int e^x \, dx \right\} dx$

$$= x^2 e^x - \int \{2x e^x\} dx$$

$$= x^2 e^x - 2 \int x e^x \, dx$$

Again by parts method

$$= x^2 e^x - 2\left[\int e^x dx - \int \left\{ \frac{d}{dx} x \int e^x \, dx \right\} dx \right]$$

$$= x^2 e^x - 2\left[xe^x - \int (1e^x) dx \right]$$

$$= x^2 e^x - 2xe^x + e^x + c$$

EXERCISE -VI

1. $\int x \cos x \, dx$
2. $\int \log(x+1) dx$
3. $\int x e^x \, dx$
4. $\int x^2 \log x \, dx$
5. $\int x \cot^2 x \, dx$
6. $\int x^2 \cos 2x dx$
7. $\int x^2 e^{-x} dx$
8.

ANSWERS

1. $x \sin x + \cos x + c$
2. $x \log(x + 1) - x + \log(x + 1) + c$
3. $xe^x - e^x + c$
4. $\frac{x^3}{3} \log x - \frac{1}{9} x^3 + c$
5. $-x \cot x + \log \sin x - \frac{x^2}{2} + c$
6. $\frac{x^2}{2} \sin 2x + \frac{x}{2} \cos 2x - \frac{\sin 2x}{4} + c$
7. $-x^2 e^{-x} - 2xe^{-x} - 2e^{-x} + c$
8. $-x \cot x + \log \sin x + c$

5.3EVALUATION OF DEFINITE INTEGRALS

Introduction

If f(x) is a continuous function defined on closed interval [a, b] and F(x) is the integral of f(x) i.e.

$$\int f(x) \, dx = F(x)$$

then definite integral of f(x) in closed interval [a, b] is

$= \frac{2}{3}[27 - 8] = 2 \times \frac{19}{3} = \frac{38}{3}$

(v) $\int_1^2 \frac{1}{x}\,dx = [\log x]_1^2 = \log 2 - \log 1 = \log 2$ as $(\log 1 = 0)$

Example 10. Evaluate (i) $\int_0^1 \frac{dx}{1+x^2}$ (ii) $\int_1^2 \frac{dx}{x\sqrt{x^2-1}}$ (iii) $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

Sol : (i) $\int_0^1 \frac{dx}{1+x^2} = [\tan^{-1} x]_0^1 = \tan^{-1} 1 - \tan^{-1} 0$

$= \tan^{-1} \tan\frac{\pi}{4} - \tan^{-1} \tan 0 = \left(\frac{\pi}{4} - 0\right) = \frac{\pi}{4}.$

(ii) $\int_1^2 \frac{dx}{x\sqrt{x^2-1}} = [\sec^{-1} x]_1^2$

$= \sec^{-1} 2 - \sec^{-1} 1$

$$= \sec^{-1}\sec\frac{\pi}{3} - \sec^{-1}\sec 0$$

$$= \frac{\pi}{3} - 0 = \frac{\pi}{3}$$

EXERCISE -VII

1. Find the value of the integral .

   a. b. c. d. 2

2. Evaluate the integral

   a. b. c. d.

3. Evaluate : (i) $\int_2^3 \frac{dx}{x}$ (ii) $\int_{-5}^{5} x\,dx$ (iii) $\int_1^2 \frac{dx}{1+3x}$

(iv) $\int_2^3 (x^2+1)dx$ (v) $\int_{-2}^{2} e^{-x}\,dx$ (vi) $\int_0^{\pi/3} \cos x\,dx$

(vii) $\int_0^2 \frac{dx}{4+3x}$ (viii) $\int_0^2 \frac{dx}{x^2+4}$ (ix) $\int_0^{\pi/4} \tan x dx$

ANSWERS

1. (c) 2. (a)

3. (i) $\log\frac{3}{2}$ (ii) 0 (iii) $\frac{1}{3}\log\frac{7}{4}$ (iv) $\frac{22}{3}$ (v) $-e^{-2} + e^{2}$

(vi) $\frac{\sqrt{3}}{2}$ (vii) $\frac{1}{3}\log\frac{5}{2}$ (viii) $\frac{\pi}{8}$ (ix) $\log\sqrt{2}$.

Integral of the type when limit is 0 to $\frac{\pi}{2}$

$$\int_0^{\pi/2} \sin^n x\,dx \quad \text{or} \quad \int_0^{\pi/2} \cos^n x\,dx \quad \text{or} \quad \int_0^{\pi/2} \sin^n x \cos^m x\,dx$$

where n, m are natural numbers, we have

(i)

(ii)

(iii) $$\int_0^{\pi/2} \sin^n x \cos^m x\,dx = \frac{(n-1)(n-3)\ldots(m-1)(m-3)\ldots}{(n+m)(n+m-2)(n+m-4)\ldots} \left(\times\frac{\pi}{2}\right. \text{ if n and m both are even)}$$

Note : Above formulae are called 'Reduction Formulae'

Example 11. Evaluate (i) $\int_0^{\pi/2} \sin^5 x\,dx$ (ii) $\int_0^{\pi/2} \cos^4 x\,dx$ (iii) $\int_0^{\pi/2} \sin^3 x \cos^2 x\,dx$

Sol : (i) $\int_0^{\pi/2} \sin^5 x\,dx = \frac{4\times 2}{5\times 3\times 1} = \frac{8}{15}$

(ii) $\int_0^{\pi/2} \cos^4 x\,dx = \frac{3\times 1}{4\times 2}\times\frac{\pi}{2} = \frac{3\pi}{16}$

Here n = 4(even)

(iii) $\int_0^{\pi/2} \sin^3 x \cos^2 x\,dx = \frac{2\times 1}{5\times 3\times 1} = \frac{2}{15}$

EXERCISE-VIII

Evaluate the integral :

1. $\int_0^{\pi/2} \sin^8 x\,dx$
2. $\int_0^{\pi/2} \cos^5 x\,dx$
3. $\int_0^{\pi/2} \sin^4 x\,dx$
4. $\int_0^{\pi/2} \cos^6 x\,dx$
5. $\int_0^{\pi/2} \cos^3 x \sin^4 x\,dx$
6. $\int_0^{\pi/2} \cos^6 x \sin^4 x\,dx$

7. Evaluate $I = \dfrac{\int_0^{\pi/2} \sin^3 x \cos^5 x \, dx}{\int_0^{\pi/2} \cos^4 x \, dx}$

ANSWERS

1. $\frac{35\pi}{256}$ 2. $\frac{8}{15}$ 3. $\frac{3\pi}{16}$ 4. $\frac{5\pi}{32}$ 5. $\frac{2}{35}$

6. $\frac{3\pi}{512}$ 7. $\frac{2}{9\pi}$

5.4 APPLICATIONS OF INTEGRATION

There are many applications that requires integration techniques like Area under the curve, volumes of solid generated by revolving the curve about axes, velocity, acceleration, displacement, work done, average value of function. But in this chapter, we will be taking a look at couple of applications of integrals.

(i) If y = f(x) is any function of x, then

Area under the curve is $\int_{x=a}^{x=b} y \, dx$

(ii) If v = f(t) is velocity of particle at time t then,

Displacement $S = \int_{t=t_1}^{t=t_2} v \, dt$

Example 12. Find the area under the curve $y = 1 + 2x^3$, when $0 \le x \le 2$.

Sol : Given $y = 1 + 2x^3$, a = 0, b = 2.

$$\text{Area} = \int_a^b y \, dx = \int_0^2 (1 + 2x^3) dx$$

$$= \left[x + \frac{x^4}{2} \right]_0^2 = (2 + 8) - (0 + 0) = 10 \text{ units}$$

Example 13. Find area under the curve of y = sin x, when $0 \le x \le \frac{\pi}{2}$.

Sol : Equation of curve y = sin x,   a = 0, b = $\frac{\pi}{2}$

$$\text{Area} = \int_a^b y\,dx = \int_0^{\pi/2} \sin x\,dx = [-\cos x]_0^{\pi/2}$$

$$= [-\cos \tfrac{\pi}{2}] - [-\cos 0]$$

$$= [-0] + 1 = 1 \quad \text{Square units}$$

EXERCISE-IX

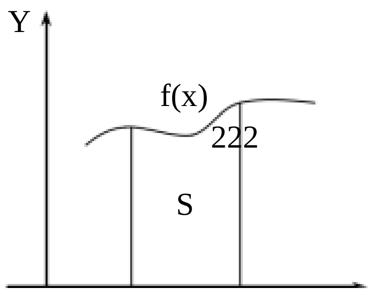
1. Find the area under the curve $y = 4x^2$, when $0 \le x \le 3$.
2. Find the area under the curve $y = \frac{1}{x}$, when $2 \le x \le 4$.
3. Find the area under the curve y = cos x, when $0 \le x$, $\frac{\pi}{4}$.
4. Find the area under the curve $y = e^{2x}$, when $0 \le x \le 1$.
5. Find area under the curve Yy = $\sqrt{2x + 3}$, when $3 \le x \le 11$.

ANSWERS

1. 36   2. log 2   3. $\frac{1}{\sqrt{2}}$ 4. $\frac{e^2 - 1}{2}$ 5. $\frac{98}{3}$

5.5NUMERICAL INTEGRATION OR APPROXIMATE INTEGRATION

Numerical integration is an approximate solution to a definite integral $\int_a^b f(x)\,dx$. In this chapter we are taking two methods to find approximate value of definite integral, $\int_a^b f(x)\,dx$ as area under the curve when $a \le x \le b$.



(i) Trapezoidal Rule

(ii) Simpson"s Rule

(i) Trapezoidal Rule : If y = f(x) be the equation of curve, then approximate area under the curve y = f(x), when $a \le x \le b$ is

$$\int_a^b y\,dx = \frac{h}{2}\left[(\text{first ordinate} + \text{last ordinate}) + 2\,(\text{sum of remaining ordinates})\right]$$

Here $h = \frac{b-a}{n}$

$a \to$ lower limit of x

$b \to$ upper limit of x

$n \to$ Number of integrals

* Ordinates are always one more than the number of intervals.

In mathematics, and more specifically in Number analysis, the trapezoidal rule (also known as trapezoid rule or trapezium rule) is a technique, for approximating the definite integral $\int_a^b f(x)\,dx$. The trapezoidal rule works by approximating the region under the graph of function f(x) as a trapezoid and calculating its area.

Example 14. Find the approximate area under the curve using trapezoidal rule, determined by the data given below

x	0	1	2	3	4	5
y	0	2.5	3	4.5	5	7.5

Sol : Here equation of curve not given so let y = f(x)

Lower limit of x, i.e. a = 0

Upper limit of x , i.e. b = 5

Gap between values of x i.e. h = 1

Number of ordinates in table are = 6

So by trapezoidal Rule

$$\int_0^5 y\,dx = \frac{h}{2}[(y_1 + y_6) + 2(y_2 + y_3 + y_4 + y_5)]$$

$y_1 = 0$, $y = 2.5$, $y_3 = 3$, $y_4 = 4.5$, $y_3 = 5$, $y_6 = 7.5$

$$= \frac{1}{2}[(0 + 7.5) + 2(2.5 + 3 + 4.5 + 5)]$$

$$= \frac{1}{2}[7.5 + 30] = 18.75 \text{ square units}$$

Example 15. Using trapezoidal rule, evaluate $\int_0^{2.5} (1 + x)\,dx$ by taking six ordinates.

Sol : Given $y = 1 + x$, $a = 0$, $b = 2.5$

as ordinates are 6 $\Rightarrow n = 5$

$$h = \frac{b - a}{n} = \frac{2.5 - 0}{5} = 0.5$$

x	0	0.5	1	1.5	2	2.5
y = 1 + x	1	1.5	2	2.5	3	3.5

Here $y_1 = 1$, $y_2 = 1.5$, $y_3 = 2$, $y_4 = 2.5$, $y_5 = 3$, $y_6 = 3.5$.

By trapezoidal Rule

$$\int_0^{2.5} (1 + x)\,dx = \frac{h}{2}[(y_1 + y_6) + 2(y_2 + y_3 + y_4 + y_5)]$$

$$= \frac{0.5}{2}[(1 + 3.5) + 2(1.5 + 2 + 2.5 + 3)]$$

$$= \frac{1}{4}[4.5 + 18] = \frac{22.5}{4} = 5.6 \text{ square units}$$

EXERCISE -X

1. Using trapezoidal rule, evaluate $\int_0^2 \sqrt{(9 - x^2)}\,dx$ by taking 4 equal intervals.

2. Using trapezoidal Rule, Find the approximate area under the curve $y = x^2 + 1$, when $0 \le x \le 6$ by taking 7 ordinates.

3. A curve is drawn to pass through the points given below

x	2	2.2	2.4	2.6	2.8	3.0
y	3	3.4	3.7	3.9	4	3.2

Find the approximate area bounded by the curve, x axis and the lines x = 2 and x = 3.

4. Apply Trapezoidal rule to evaluate $\int_4^8 \frac{1}{x+2}$ by taking 4 equal intervals.

ANSWERS

1. 5.4 2. 79 3. 3.62 4. 0.5123

Simpson's $\frac{1}{3}^{rd}$ Rule : In this rule, the graph of curve y = f(x) is divided into 2n (Even) intervals.

Let y = f(x) be the equation of curve. When $a \le x \le b$, then approximate area under the curve is

$$\int_a^b y\,dx = \frac{h}{3}\,[(\text{first ordinate} + \text{last ordinate}) + 2(\text{sum of remaining odd ordinates})$$

$$+ 4(\text{sum of remaining even ordinates})]$$

where $h = \frac{b-a}{n}$

$a \to$ lower limit of x

$b \to$ upper limit of x

$n \to$ number of intervals

Note : Simpson"s Rule is valid when total number of ordinates are odd.

Example 16. Calculate by Simpson"s rule an approximate value of $\int_2^8 (x^3 + 3)dx$ by taking 7 ordinates.

Sol : Given $y = x^3 + 3$, a = 2, b = 8, n = 6

$$\therefore \quad h = \frac{b-a}{n} = \frac{8-2}{6} = 1$$

x	2	3	4	5	6	7	8
$y = x^3 + 3$	11	30	67	128	219	347	515

Here $y_1 = 11$, $y_2 = 30$, $y_3 = 67$, $y_4 = 128$, $y_5 = 219$, $y_6 = 347$, $y_7 = 515$.

By Simpson"s rule

$$\int_2^8 (x^3 + 3)dx = \frac{h}{3}(y_1 + y_7) + 2(y_3 + y_5) + 4(y_2 + y_4 + y_6)]$$

$$= \frac{1}{3}[(11 + 515) + 2(67 + 219) + 4(30 + 128 + 347)]$$

$$= \frac{1}{3}[526 + 572 + 2020] = \frac{1}{3}[3118]$$

= 1039.3 square units.

Example 17. Evaluate $\int_0^4 e^x dx$ by Simpson"s rule, when $e = 2.72$, $e^2 = 7.39$, $e^3 = 20.09$, $e^4 = 54.60$.

Sol. Given $y = e^x$, $a = 0$, $b = 4$, $h = 1$(As values of x varies in gap of 1).

x	0	1	2	3	4
$Y = e^x$	$e^0 = 1$	$e^1 = 2.72$	$e^2 = 7.39$	$e^3 = 20.09$	$e^4 = 54.6$

$y_1 = 1$, $y_2 = 2.72$, $y_3 = 7.39$, $y_4 = 20.09$, $y_5 = 54.6$

By Simpson"s Rule

$$\int_0^4 e^x dx = \frac{h}{3}[(y_1 + y_5) + 2y_3 + 4(y_2 + y_4)]$$

$$= \frac{1}{3}[(1 + 54.6) + 2 \times 7.39 + 4(2.72 + 20.09)]$$

$$= \frac{1}{3}[55.6 + 14.78 + 91.24] = 53.87 .$$

EXERCISE - XI

1. Calculate by Simpson‟s rule an approximate value of $\int_0^1 \frac{dx}{1+x} dx$ by taking 10 equal interval.

2. Apply Simpson‟s rule to evaluate approximate value of $\int_4^8 \frac{dx}{x-3}$ by taking four equal intervals.

3. Find approximate area under the curve $y = (1 + x^2)$ when $0 \leq x \leq 4$ using Simpson‟s Rule by taking five ordinates.

4. Use Simpson‟s rule to approximate the area under the curve determined by the data gien below

x	1	2	3	4	5	6	7
y	15	27	30	33	20	19	4

ANSWERS

1. . 6931 2. 1.62 3. 25.33 4. 145

UNIT – 6

DIFFERENTIAL EQUATION

Learning Objectives

- To understand the basic concept about differential equations.
- To learn the solving technique for first order differential equations.

6.1 DIFFERENTIAL EQUATION

In engineering problems, mathematical models are made to represent certain problems. These mathematical models involve variables and derivatives of unknown functions. Such equations form the differential equation. Thus, a Differential Equation is defined as an equation in which y and its derivatives exists in addition to the independent variable x.

For example; (i) $\frac{dy^2}{dx} + 5y = 0$ (ii) $\frac{dy}{dx} + 3xy = 0$ (iii) $\left(\frac{d^2y}{dx}\right)^2 = 1 + \frac{dy}{dx}$

In this chapter we will study only about ordinary differential equation.

Ordinary differential Equation: A differential equation in which dependent variable „y" depends on only one variable „x" i.e., y = f(x) . e.g.

$$\frac{dy}{dx} + y = 0$$

Partial differential Equation : A differential equation in which dependent variable „z" depends on more than one variable say x and y i.e., z = f(x, y), For example,

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

Order of differential equation : It is the order of the highest derivative appearing in the differential equation.

Example 1. Find the order (i) $\frac{d^2y}{dx^2} + 5y = 0$ (ii) $\frac{d^3y}{dx^3} + 2\left(\frac{d^2y}{dx^2}\right)^2 + 3y = 0$

Sol. (i) order is 2 (ii) order is 3

Degree of Differential Equation: It is the power of highest derivative appearing in a differential equation when it is free from radicals as far as derivatives are concerned fractional and negative powers.

Example 2. (i) $\left(\frac{d^2y}{dx^2}\right)^3 + y = 0$ (ii) $\frac{d^2y}{dx^2} = \sqrt{1+\left(\frac{dy}{dx}\right)^3}$

Sol. Order – 2

degree – 3

Squaring both sides to remove fractional powers

$$\left(\frac{d^2y}{dx^2}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^3$$

Now order → 2, degree → 2

(iii) now order → 2, degree → 3

Linear Differential Equation : A differential equation in which dependent variable „y" and its derivatives has power one and they are not multiplied together. When y = f(x)

Ex $$\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + 6\frac{dy}{dx} + y = 0$$

First Order Linear Differential Equation

A Differential Equation of the form

$$\frac{dy}{dx} + P(x)\,y = 0 \qquad \text{where } y = f(x)$$

EXERCISE -I

1. Define the following with Examples

   (i) Differential equation

   (ii) Order of Differential Equation

   (iii) Degree of Differential Equation

2. Find the order and degree of the differential equation $\frac{d^3y}{dx^3} + \sqrt{\frac{d^2y}{dx^2}} + y = x$

   a. Order= 3, degree= 3 c. Order= 2, degree= 3 b. Order= 3, degree= 2 d. Order= 3, degree= 1

3. Find the degree of the differential equation

   a. 1 b. 2 c. 4 d. None of these

4. Find the degree of the differential equation $(x + 1) = \left(\frac{d^2y}{dx^2}\right)^{\frac{2}{3}}$

   a. b. 3 c. 2 d. 1

5. Determine the order and degree of the following differential equation. State whether the equations is linear or non linear

(i) $\left(\frac{d^3y}{dx^3}\right)^2 + 7\left(\frac{d^2y}{dx^2}\right)^{10} + 9\frac{dy}{dx} + 7y = 0$

(ii) $\frac{dy}{dx} + \cos x = 0$

(iii) $xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y = 0$

(iv) $\frac{d^2y}{dx^2} + \log x\frac{dy}{dx} + y^2 = 0$

(v) $\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^2 + 5y = 0$

(vi) $\left(\frac{d^2y}{dx^2}\right)^{\frac{1}{3}} = \left(\frac{dy}{dx}\right)^{\frac{1}{2}}$

(vii) $\frac{d^4y}{dx^4} + 3\frac{dy}{dx} = 1$

(viii) $\left[1+\left(\frac{d^2y}{dx^2}\right)^2\right]^{\frac{1}{3}} = \frac{d^3y}{dx^3}$

ANSWERS

2.(b) 3. (a) 4. (c)

5.	(i) Order – 3	degree –2	non linear
	(ii) Order – 1	degree – 1	Linear
	(iii) Order – 2	degree – 1	non linear
	(iv) Order – 2	degree – 1	non linear
	(v) Order – 3	degree – 1	non linear
	(vi) Order – 2	degree – 2	non linear
	(vii) Order – 4	degree – 1	linear
	(viii) Order – 3	degree – 3	non linear

Solution of a Differential Equation

The solution of differential equation is the relation between the variables involves in differential equation which satisfies the given differential equation.

In this chapter we will study the solution of first order differential equation by variable separable method.

Steps :

1. Separate the variables in given differential equation keeping in mind that dy and dx should be in numerator.

2. Now put the sign of $\int$ on both sides.

3. Integrate both sides separately by adding a constant on one side

4. This will give us general sol of Differential equation.

Example 3. : Find the general solution of differential equation $\frac{dy}{dx} = \sin x$.

Sol : Given $\frac{dy}{dx} = \sin x$

Separate the variables, we get

$$dy = \sin x \, dx$$

Put the sign of $\int$ on both sides

$$\int dy = \int \sin x \, dx$$

$$y = -\cos x + C$$

Example 4. Find the general solution of differential equation $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$.

Sol : Separate the variables $\frac{dy}{1+y^2} = \frac{dx}{1+x^2}$

Put the sign of $\int$ on both sides

$$\int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2}$$

$$\tan^{-1} y = \tan^{-1} x + c$$

Example 5. Find the solution of differential equation :

(i) $\frac{dy}{dx} = \frac{x}{y}$ (ii) $\frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}$

(iii) $x dy - y\, dx = 0$ (iv) $\frac{dy}{dx} = \frac{1+y}{1+2x}$

Sol : (i) $\frac{dy}{dx} = \frac{x}{y} \Rightarrow y dy = x\, dx$

$$\Rightarrow \int y\, dy = \int x\, dx \quad \Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + c$$

(ii) $\frac{dY}{dx} = \frac{1}{\sqrt{1+x^2}} \Rightarrow dy = \frac{dx}{\sqrt{1+x^2}}$

$$\Rightarrow \int dy = \int \frac{dx}{\sqrt{1+x^2}}$$

$$\Rightarrow \quad y = \log|x + \sqrt{1+x^2}| + c$$

(iii) x dy – y dx = 0 ⇒ x dy = y dx

$$\Rightarrow \frac{dy}{y} = \frac{dx}{x}$$

$$\Rightarrow \int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\Rightarrow \log y = \log x + c$$

(iv) $\frac{dy}{dx} = \frac{1+y}{1+2x} \quad \Rightarrow \frac{dy}{1+y} = \frac{dx}{1+2x}$

$$\Rightarrow \int \frac{dy}{1+y} = \int \frac{dx}{1+2x}$$

$$\Rightarrow \log(1+y) = \frac{\log(1+2x)}{2} + c$$

EXERCISE-II

1. Solve the differential equation

   a. c.

   b. d.

2. The general sol of the differential equation is ....

   a. c.

   b. =c d. None of these

3. The general sol of the differential equation $\frac{dy}{dx} = e^x . e^{-y}$ is

   a. $e^y = e^x + c$ c. $e^{-y} = e^x + c$ b. $e^x + e^{-y} = c$ d. $e^x e^y = c$

4. Solve the differential equations

(i) $\frac{dy}{dx} = \frac{x+2}{y}$ (iv) $\frac{dy}{dx} = \sqrt{\frac{1+y^2}{1+x^2}}$

(ii) $\frac{dy}{dx} = \frac{2 - y}{x + 1}$

(v) $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$

(iii) $\frac{dy}{dx} = y \tan 2x$

(vi) $(1 - x)y\,dx + (1 + y)x\,dy = 0$

5. Solve the differential equation $\frac{dy}{dx} = 1 - x + y - xy$.

6. Solve the differential equation $\sin x \cos y\, dx + \cos x \sin y\, dy = 0$

7. Solve the differential equation $\frac{dy}{dx} = 2e^{x} y^3$.

ANSWERS

1. (c) 2. (b) 3. (a)

4. (i) $\frac{y^2}{2} = \frac{x^2}{2} + 2x + c$

(ii) $\frac{\log(2 - y)}{-1} = \log(x + 1) + c$

(iii) $\log y = \frac{\log \sec 2x}{2} + c$

(iv) $\log | y + \sqrt{1 + y^2} | = \log | x + \sqrt{1 + x^2} | + c$

(v) $e^{y} = e^{x} + \frac{x^3}{3} + c$

(vi) $\log xy = x - y + c$

5. $\log(1 + y) = x - \frac{x^2}{2} + c$

6. $\log \sec x + \log \sec y = c$

7. $-\frac{y^{-2}}{2} = 2e^{x} + c$

UNIT– 7
STATISTICS

UNIT– 7
STATISTICS

Learning Objectives

- To understand the basic statistical measures like mean, mode, median and their different calculation methods.
- To learn about methods to calculate measure of mean derivatives, standard deviation, variance etc.
- To understand concept of correlation of data and different methods to calculate coefficients of correlation.

STATISTICS

Statistics is a branch of mathematics dealing with collection of data, analysis, interpretation, presentation and organization of data. In applying statistics into a problem, to for Example a scientific, industrial or social problem, it is conventional to begin with statistical population or a statistical model process to be studied in that problem.

Here in this chapter, we study about some measures of central tendency which are a central or typical values for a probability distribution. In other words, measures of central tendency are often called averages. Before we study about measures of central tendency, we must know about type of data.

Type of Data

a) Raw data : Data collected from source without any processing.

For example:The marks of ten students in math subject are 10,12,15,25,30,25,15,13,20,25

b) Discrete frequency distribution (Ungrouped data): The data which have not been divided into groups and presented with frequency (no. of times appeared) of data.

For example:The marks of ten students in math subject are given by

Marks(x):	10	12	13	15	20	25	30
Frequency(f):	1	1	1	2	1	3	1

c) Continuous frequency distribution (Grouped data): The data which have been divided into continuous group and presented with frequency (no. of times appeared) of individual group.

For example:The marks of students in math subject in a class are given by:

Marks class	10-20	20-30	30-40	40-50
Frequency	7	10	15	8

7.1 MEASURE OF CENTRAL TENDENCY

Measure of Central tendency are also classed as summary of statistics. In general,the mean, median and mode are all valid measure of central tendency.

Mean: It is average value of given data.

For Raw data;**Mean** $\bar{x} = \frac{\sum x_i}{n}$; where x_i is value of individual data

For Ungrouped and Grouped data; **Mean** $\bar{x} = \frac{\sum f_i x_i}{\sum f_i}$; where n_i is the value of individual data and f_i is corresponding frequency of value x_i.

Median : It is middle value of arranged data.

For Grouped data; **Median =** $l + \frac{\left(\frac{n}{2} - c\right)}{f} * h$ where median class l = lower limit of class which has corresponding cumulative frequency equal to or greater than $\frac{N}{2}$ median class,

$n = \Sigma f_i$,

c = Comulative frequency preceding to the median class cumulative frequency h = length of interval,

f = frequency of median class

Mode : It is the term which appears maximum number of times in the given data.

OR

The term which has highest frequency in given data.

For Grouped data;**Mode =** $l + \frac{(f - f_1)}{2f - f_1 - f_2} * h$

Modal class = class which has maximum frequency value

where l = lower limit of modal class, h = length of interval,

f = frequency of modal class

f_1 = frequency of preceding class to modal class f_2

= frequency of succeeding class to modal class

Example 1. Calculate mean for the following data.

21, 23, 25, 28, 30, 32, 46, 38, 48, 46

Sol : The given data is 21, 23, 25, 28, 30, 32, 46, 38, 48, 46

Total No. of observations = 10

Sum of observations = 21 + 23 + 25 + 28 + 30 + 32 + 46 + 38 + 48 + 46 = 337

$\therefore$ Mean = $\frac{Sum\ of\ observations}{Total\ No.of\ observations} = \frac{337}{10} = 33.7$

Example 2. Find the arithmetic mean of first 10 natural numbers.

Sol : First 10 natural numbers are 1,2,3,4,5,6,7,8,9,10

Sum of numbers = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55

No. of terms = 10

$\therefore$ Mean = $\frac{sum\ of\ numbers}{No.of\ terms} = \frac{55}{10} = 5.5$

Example3. Find the median of the daily wages of ten workers.

(Rs.) : 20, 25, 17, 18, 8, 15, 22, 11, 9, 14

Sol : Arranging the data in ascending order, we

have 8, 9, 11, 14, 15, 17, 18, 20, 22, 25

Since there are 10 observations, therefore median is the arithmetic mean of $\left(\frac{10}{2}\right)^{th}$ and $\left(\frac{10}{2}+1\right)^{th}$ observations, so median = $\frac{15+17}{2} = 16$

Example 4. The following are the marks of 9 students in a class. Find the median. 34, 32, 48, 38, 24, 30, 27, 21, 35

Sol : Arranging the data in ascending order, we have 21, 24, 27, 30, 32, 34, 35, 38, 48

Since there are total 9 number of terms, which is odd. Therefore, median is the value of $\left(\frac{9+1}{2}\right)^{th}$ observation *i.e.* 32

Example 5. Find the mode from the following data:

110, 120, 130, 120, 110, 140, 130, 120, 140, 120

Sol : Arranging the data in the form of a frequency table, we have

Value:	110	120	130	140
Frequency :	2	4	2	2

Since the value 120 occurs the maximum number of times. Hence the mode value is 120 **Example 6.** The arithmetic mean of 7, 9, 5, 2, 4, 8, x is given to be 7. Find x.

Sol : $\bar{x} = \frac{7+9+5+2+4+8+x}{7}$

but $\bar{x} = 7$

$$7 = \frac{35+x}{7}$$

$\Rightarrow$ 49 = 35 + x

$\Rightarrow$ x = 49 – 35 = 14

Example 7. Calculate range for the following data

19, 25, 36, 72, 51, 43, 28

Sol : Maximum value of given data = 72

Minimum value of given data = 19

Range of data = Maximum value – Minimum value

= 72 – 19 = 53

Example 8. Calculate arithmetic mean for the following :

Income (in Rs.)	:	500	520	550	600	800	1000
No. of Employees	:	4	10	6	5	3	2

Income (in Rs.) (x_i)	No. of Employees (f_i)	$(f_i x_i)$
500	4	2000
520	10	5200
550	6	3300
600	5	3000
800	3	2400
1000	2	2000
	$\Sigma f_i = 30$	$\Sigma x_i f_i = 17900$

$$\therefore \quad \bar{x} = \frac{\sum x_i f_i}{\sum f_i} = \frac{17900}{30} = 590.67$$

Example 9. Calculate median for the following data

x_i	:	1	2	3	4	5	6	7	8	9
f_i	:	8	10	11	16	20	25	15	9	6

Sol :

x_i	f_i	C.f
1	8	8
2	10	18
3	11	29
4	16	45

Here N = 120

Now $\frac{N}{2} = \frac{120}{2} = 60$

We find that the commutative frequency justgreater than $\frac{N}{2}$, is

5	20	65	65 and the value of x corresponding to 65 is „5‟. Therefore median is „5‟.
6	25	90	
7	15	105	
8	9	114	
9	6	120	
	=N = 120 Σf_i		

Example 10. Calculate mode for the following data:

x_i	91	92	96	97	101	103	108
f_i	3	2	3	2	5	3	3

Sol : As we know that mode is the highest frequency value and highest frequency is 5 and corresponding value is 101.

So mode value is 101

Example 11. Calculate mean for the following frequency distribution

Class interval	:	0-8	8-16	16-24	24-32	32-40	40-48
Frequency	:	8	7	16	24	15	7

Sol :

Class interval	Frequency(f_i)	Mid Value (x_i)	f_ix_i
0-8	8	4	32
8-16	7	12	84
16-24	16	20	320
24-32	24	28	672
32-40	15	36	540
40-48	7	44	308
	$\Sigma f_i = 77$		$\Sigma x_if_i = 1956$

$$\bar{x} = \frac{\sum x_i f_i}{\sum f_i} = \frac{1956}{77} = 25.40$$

Example 12. Calculate the median from the following distribution

Class :	5-10	10-15	15-20	20-25	25-30	30-35	35-40	40-45
Fre. :	5	6	15	10	5	4	2	2

Sol :

Class	f_i	C.f
5-10	5	5
10-15	6	11
15-20	15	26
20-25	10	36
25-30	5	41
30-35	4	45
35-40	2	47
40-45	2	49
	$\Sigma f_i = N = 49$	

We have N =49

Now $\frac{N}{2} = \frac{49}{2} = 24.5$

commutative frequency just greater than $\frac{N}{2}$, is 24.5 and Corresponding class is 15-20

Thus, class 15-20 is the median class such that l = 15, f = 15

c.f. = 11, h = 5

$$\therefore \text{ Median} = l + \frac{\left(\frac{N}{2} - c\right)}{f} \times h = 15 + \frac{(24.5 - 11)}{15} \times 5 = 15 + \frac{13.5}{3} = 15 + 4.5 = 19.5$$

Example 13. Calculate mode from the following data

Rent(in Rs.) :	20-40	40-60	60-80	80-100	100-120	120-140	140-160
No. of House :	6	9	11	14	20	15	10

Sol : By observation, we find that the highest frequency is 20

Hence 100-120 is the modal class.

$$\text{Mode} = l + \frac{(f - f_1)}{(2f - f_1 - f_2)} \times h$$

where l = 100, f = 20, f_1 = 14, f_2 = 15, h = 20

$\therefore$ Mode = $100 + \frac{(20-14)}{2(20)-14-15} \times 20$

$$= 100 + \frac{6 \times 20}{11} = 100 + \frac{120}{11} = 100 + 10.91 = 110.91$$

7.2 MEASURE OF DISPERSION

Mean Deviation : The mean deviation is the first measure of dispersion. It is the average of absolute differences between each value in a set of value, and the average value of all the values of that set. Mean Deviation is calculated either from Mean or median. Here, we will study Mean Deviation about Mean

1. Mean deviation about Mean

(a) For Raw data

Mean Deviation about Mean = $\frac{\sum |x_i - \bar{x}|}{N}$

Where $\Sigma \rightarrow$ Represents summation

$N \rightarrow$ Number of terms or observations

Mean $\bar{x} = \frac{\sum x_i}{N}$

(b) For Ungrouped and Grouped data

Mean Deviation about Mean = $\frac{\sum f_i |x_i - \bar{x}|}{\sum f_i}$

Where $\Sigma \rightarrow$ Represents summation

$f_i \rightarrow$ frequency

$\bar{x} = \frac{\sum f_i x_i}{\sum f_i}$

Also Coefficient of Mean Deviation about mean is = $\frac{\text{Mean deviation about mean}}{\text{Mean}}$

2. Standard Deviation

It is the Root Mean Square value of deviations and also called in short form as R.M.S. value.

(a) For Raw data

$$\sigma = \text{Standard Deviation} = \sqrt{\frac{\sum (x_i - \bar{x})^2}{N}}$$

where $\bar{x} = \frac{\sum x_i}{N}$

N → Number of observations

(b) For Ungrouped and Grouped data

$$\sigma = \text{Standard Deviation} = \sqrt{\frac{\sum f_i (x_i - \bar{x})^2}{\sum f_i}}$$

where $\bar{x} = \frac{\sum f_i x_i}{\sum f_i}$

Some formulae :

(i) Co-efficient Standard Deviation = $\frac{\text{Standard Deviation}}{\bar{x}}$

(ii) Variance = $(\sigma)^2$ = Square of Standard Deviation

(iii) Co-efficient of Variation = $\frac{S.D}{\bar{x}} \times 100$

Example 14. Calculate mean deviation about mean and its coefficient from the following data

21, 23, 25, 28, 30, 32, 46, 38, 48, 46

Sol:

x_i	$x_i - \bar{x}$	$\lvert x_i - \bar{x} \rvert$
21	– 12.7	12.7
23	–10.7	10.7
25	–8.7	8.7
28	–5.7	5.7
30	–3.7	3.7
32	–1.7	1.7
46	12.3	12.3
38	4.3	4.3
48	14.3	14.3
46	12.3	12.3
$\Sigma x_i = 337$		$\Sigma \lvert x_i - \bar{x} \rvert = 86.4$

$$\bar{x} = \frac{\Sigma x_i}{N} = \frac{337}{10} = 33.7$$

$$M.D. = \frac{\sum |x_i - \bar{x}| f_i}{\sum f_i} = \frac{86.4}{10} = 8.64$$

$$\text{Coefficient of M.D.} = \frac{M.D.}{\bar{x}} = \frac{8.64}{33.7} = 0.26$$

Example 15. Find mean deviation about mean of the following data

x_i	:	3	5	7	9	11	13
f_i	:	2	7	10	9	5	2

Sol:

x_i	f_i	$x_i f_i$	$x_i - \bar{x}$	$\|x_i - \bar{x}\|$	$\|x_i - \bar{x}\| f_i$
3	2	6	–4.8	4.8	9.6
5	7	35	–2.8	2.8	19.6
7	10	70	–0.8	0.8	8.0
9	9	81	1.2	1.2	10.8
11	5	55	3.2	3.2	16.0
13	2	26	5.2	5.2	10.4
	$\Sigma f_i = 35$	$\Sigma x_i f_i = 273$			$\Sigma\|x - \bar{x}\| f_i = 74.4$

$$\bar{x} = \frac{\sum x_i f_i}{\sum f_i} = \frac{273}{35} = 7.8 \qquad M.D. = \frac{\sum |x_i - \bar{x}| f_i}{\sum f_i} = \frac{74.4}{35} = 2.13$$

Example 16. Find mean deviation about the mean for the following data

Marks obtained	10-20	20-30	30-40	40-50	50-60	60-70	70-80
Number of students	2	3	8	14	8	3	2

Sol :

Marks obtained	No. of students(f_i)	Mid point(x_i)	$f_i x_i$	$x_i - \bar{x}$	$\|x_i - \bar{x}\|$	$\|x_i - \bar{x}\| f_i$
10-20	2	15	30	–30	30	60
20-30	3	25	75	–20	20	60
30-40	8	35	280	–10	10	80
40-50	14	45	630	0	0	0
50-60	8	55	440	10	10	80
60-70	3	65	195	20	20	60
70-80	2	75	150	30	30	60

$\Sigma f_i = 40$		$\Sigma x_i f_i = 1800$			$\Sigma \lvert x_i - \bar{x} \rvert f_i = 400$

$$\bar{x} = \frac{\sum x_i f_i}{\sum f_i} = \frac{1800}{40} = 45 \qquad M.D. = \frac{\sum \lvert x_i - \bar{x} \rvert f_i}{\sum f_i} = \frac{400}{40} = 10$$

Example 17. Calculate deviation and variance of the following data :

6, 8, 10, 12, 14, 16, 18, 20, 22, 24

Sol :

x_i	$x_i - \bar{x}$	$(x_i - \bar{x})^2$
6	–9	81
8	–7	49
10	–5	25
12	–3	9
14	–1	1
16	1	1
18	2	4
20	5	25
22	7	49
24	9	81
$\Sigma x_i = 150$		$\Sigma(x_i - \bar{x})^2 = 330$

$$\bar{x} = \frac{\sum x_i}{n} = \frac{150}{10} = 15$$

$$S.D. = \sqrt{\frac{\sum (x_i - \bar{x})^2}{n}} = \sqrt{\frac{330}{10}} = \sqrt{33}$$

$$\text{Variance} = (\text{S.D.})^2 = \left(\sqrt{33}\right)^2 = 33$$

Example 18. Find variance and co-efficient of variation for the following data

x_i	4	8	11	17	20	24	32

f_i	3	5	9	5	4	3	1

Sol :

x_i	f_i	$f_i x_i$	$x_i - \bar{x}$	$(x_i - \bar{x})^2$	$f_i (x_i - \bar{x})^2$
4	3	12	–10	100	300
8	5	40	–6	36	180
11	9	99	–3	9	81
17	5	85	3	9	45
20	4	80	6	36	144
24	3	72	10	100	300
32	1	32	18	324	324
	$\Sigma f_i = 30$	$\Sigma x_i f_i = 420$			$\Sigma f_i (x_i - \bar{x})^2 = 1374$

$$\bar{x} = \frac{\sum x_i f_i}{\sum f_i} = \frac{420}{30} = 14$$

$$S.D = \sqrt{\frac{\sum f_i (x_i - \bar{x})^2}{\sum f_i}} = \sqrt{\frac{1374}{30}} = \sqrt{45.8} = 6.77$$

Variance = $(S.D.)^2$ = 45.8

Coefficientof variation = $\frac{S.D}{\bar{x}} \times 100 = \frac{6.77}{14} \times 100 = 48.35$

Example 19. Calculate the mean, variance and coff. of S.D.for the following distribution:

Class	30-40	40-50	50-60	60-70	70-80	80-90	90-100
Frequency	3	7	12	15	8	3	2

Sol :

Class	f_i	x_i	$f_i x_i$	$(x_i - \bar{x})^2$	$f_i (x_i - \bar{x})^2$
30-40	3	35	105	729	2187
40-50	7	45	315	289	2023
50-60	12	55	660	49	588
60-70	15	65	975	9	135

70-80	8	75	600	169	1352
80-90	3	85	255	529	1587
90-100	2	95	190	1089	2178
	50		3100		10050

$$\bar{x} = \frac{\sum x_i f_i}{\sum f_i} = \frac{3100}{50} = 62$$

$$\text{Variance} = \frac{\sum f_i (x_i - \bar{x})^2}{\sum f_i} = \frac{10050}{50} = 201$$

$$\text{S.D} = \sqrt{201} = 14.18$$

$$\text{Coff. of S.D} = \frac{S.D}{\bar{x}} = \frac{14.18}{62} = 0.22$$

EXERCISE – I

1. Mean of first 10 natural numbers is
   a. 5.5 b. 5 c. 4 d. 10
2. Median value of first 20 natural numbers is
   a. 10 b. 10.5 c. 11 d. 12
3. Mode of following data : 2,2,4,4,4,5,6,7,10,11 is
   a. 2 b. 4 c. 10 d. 11
4. Calculate the mean of 1, 2, 3, 4, 5, 6
5. Calculate the median of the data 13, 14, 16, 18, 20, 22
6. Median of the following observations 68, 87, 41, 58, 77, 35, 90, 55, 92, 33 is 58. If 92 is replaced by 99 and 41 by 43, then find the new median.
7. Find the mode of the set of values

   2.5, 2.3, 2.2, 2.4, 2.2, 2.7, 2.7, 2.5, 2.3, 2.3, 2.6
8. Find median of the series 3, 6, 6, 9, 12, 10
9. Find median of the series 4, 8, 6, 12, 15
10. Find A.M. for the following

x	:	10	11	12	13	14	15
f	:	2	6	8	6	2	6

11. Find the value of median for the following data

x	5	6	7	8	9	10	11	12	13	15	18	20
f	1	5	11	14	16	13	10	70	4	1	1	1

12. Find the mean deviation about mean of daily wages (in Rs.) of 10 workers : 13, 16, 15, 15, 18, 15, 14, 18, 16, 10 also find its co-efficient.

13. Find M.D. about mean of the following data

x	:	3	5	7	9	11	13
f	:	2	7	10	9	5	2

14. Find the mean deviation about the mean for the following data.

Income per day	0-100	100-200	200-300	300-400	400-500	500-600	600-700	700-800
Number of Persons	4	8	9	10	7	5	4	3

15. Find mean deviation about mean for the following data

x_i	5	10	15	20	25
f_i	7	4	6	3	5

16. Find mean deviation for 4, 7, 8, 9, 10, 12, 13, 17

17. Find standard deviation for the following data

x_i	3	8	13	18	23
f_i	7	10	15	10	6

18. Calculate mean, variance and standard deviation for the following data

Cass Interval	30-40	40-50	50-60	60-70	70-80	80-90	90-100
Frequency	3	7	12	15	8	3	2

19. Find mean and variance for 6, 7, 10, 12, 13, 4, 8, 12

20. Find variance for the following frequency distribution

Cass Interval	0-30	30-60	60-90	90-120	120-150	150-180	180-210
Frequency	2	3	5	10	3	5	2

21. Find mean for the following data

Class interval	0-10	10-20	20-30	30-40	40-50
Frequency	5	8	15	16	6

ANSWERS

(1) a	(2) b	(3) b	(4) 7/2	(5) 17	(6) 58
(7) 2.2	(8) 7.5	(9) 8	(10) 12.6	(11) 12	(12) 1.6
(13) 2.13	(14) 157.92	(15) 6.32	(16) 3	(17) 6.12	

(18) Mean; 62 Variance: 201 SD: 14.18 (19) Mean: 9 Variance: 9.25

(20) 2276 (21) 64

7.3 CORRELATION AND RANK OF CORRELATION

Correlation : If two quantities are in such way that changes in one leads to change in other then we say that these two quantities are correlated to each other. For Example, the age of a person and height of the person are correlated. If this relation exist in two variables then we say simple correlation. In this chapter, we study about simple correlation

Types of Correlation

(i) Positive Correlation : Two quantities are positively correlated if increase in one leads to increase in other or decrease in one leads to decrease in other, then we say two quantities are positively correlated. For example

Year :	1960	1970	1980	1990	2000
Population : in village	1250	1390	1490	1670	1700

(ii) Negative Correlation : Two quantities are negatively correlated if increase in one leads to decrease in other or vice versa then we say the two quantities are negatively correlated. For example

Year :	1960	1970	1980	1990	2000
No. of trees in a town :	2000	1730	1600	1520	990

Correlation Coefficient : The numerical measure of degree of relationship correlation between two quantities is called correlation coefficient.

Method of Studying Correlation

(i) Scatter Diagram Method
(ii) Graphic Method
(iii)Karl Pearson"s Coefficient of Correlation
(iv) Rank Correlation Method

Here we will study only about Rank Correlation Method.

Rank Correlation Coefficient: In this method, firstly ranking are given to the observations then, co-efficient of rank correlation is given as :

$$r = 1 - \frac{6\sum d^2}{N(N^2 - 1)}$$

where N $\rightarrow$ Number of observations

d $\rightarrow$ difference of ranks

Here r lies between -1 and 1

(i) $r > 0$ and nearly equal 1 means two quantities are positively correlated
(ii) $r < 0$ means two quantities are negatively correlated

Example 20. Calculate co-efficient of rank correlation between X and Y from the following data:

X	:	45	70	65	30	90	40	50	75	85	60
Y	:	35	90	70	40	95	45	60	80	30	50

Sol:

X	Y	R_1	R_2	$d = R_1 - R_2$	d^2
45	35	8	9	-1	1
70	90	4	2	2	4
65	70	5	4	1	1
30	40	10	8	2	4
90	95	1	1	0	0
40	45	9	7	2	4
50	60	7	5	2	4
75	80	3	3	0	0

85	30	2	10	-8	64
60	50	6	6	0	0
					$\Sigma d^2 = 82$

$$r = 1 - \frac{6\sum d^2}{n(n^2-1)} = 1 - \frac{6 \times 82}{10(99)} = 1 - \frac{492}{990} = 1 - 0.49 = 0.51$$

Example 21. In a fancy-dress competition, two judges accorded following ranks to the 10 participants:

Judge X	1	2	3	4	5	6	7	8	9	10
Judge Y	10	6	5	4	7	9	8	2	1	3

Calculate the co-efficient of rank correlation.

Sol:

Judge X = R_1	Judge Y = R_2	d = R_1-R_2	d^2
1	10	-9	81
2	6	-4	16
3	5	-2	4
4	4	0	0
5	7	2	4
6	9	3	9
7	8	-1	1
8	2	6	36
9	1	8	64
10	3	7	49
			$\Sigma d^2 = 264$

$$r = 1 - \frac{6\sum d^2}{n(n^2-1)} = 1 - \frac{6 \times 264}{10(99)} = 1 - \frac{1584}{990} = 1 - 1.6 = -0.6$$

Example 22. The co-efficient of rank correlation between X and Y is 0.143. If the sum of scourers of the differences is 48, find the value of N.

Sol : r = 0.143, $\Sigma d^2 = 48$, N = ?

$$r = 1 - \frac{6\sum d^2}{N(N^2-1)} \quad or \quad 0.143 = 1 - \frac{6 \times 48}{N(N^2-1)}$$

$$0.143 - 1 = \frac{-288}{N(N^2-1)} \quad or \quad -0.857 = \frac{-288}{N(N^2-1)}$$

$$N(N^2) - 1 = \frac{228}{0.857}$$

$N(N^2 - 1) = 336$

By hit and trial method, we get n = 7

i.e. $7(7^2 - 1) = 7(49-1) = 7(48) = 336$

So n = 7

EXERCISE – II

1. There are two sets X and Y of data. If increase in X values leads to decrease in corresponding values of Y set, then
    a. X and Y are positively correlated
    b. X and Y are negatively correlated
    c. X and Y are not correlated
    d. X and Y are equal sets
2. Formula for Rank correlation is:
    a. $r = 1 - \frac{6\sum d^2}{N(N^2-1)}$ c.
    b. $-1 \le r \le 1$ d. $r > 0$
3. The value of rank correlation coefficients satisfies
    a. $-1 \le r \le 1$ b. $r > 1$ c. $r < -1$ d.
4. Formula for mean deviation about mean for frequency data is given as

    a. b. c. d.
5. Fill in the blanks
    a. Variance is .... of standard deviation
    b. Coefficient of variation = ... × 100
6. The ranking of ten students in two subjects A and B are

A :	3	5	8	4	7	10	2	1	6	9
B :	6	4	9	8	1	2	3	10	5	7

7. Ten students got the following marks in Mathematics and Physics

Marks in Maths :	78	36	98	25	75	82	90	62	65	39
Marks in Phy. :	84	51	91	60	68	62	86	58	53	47

8. Calculate the co-efficient of rank correlation between X and Y from the following data:

X :	10	12	18	16	15	19	13	17
Y :	30	35	45	44	42	48	47	46

9. The sum of squares of differences on the ranks of n pairs of observations is 126 and the co-efficient of rank correlation is -0.5. Find n.

10. The rank correlation co-efficient between marks obtained by some students in statistics and Economics is 0.8. If the total of squares of rank difference is 33. Find the no. of students.

ANSWERS

1. b 2. a 3. a 4. a 5.a. Square b. $\frac{\text{Standard deviation}}{\bar{x}}$

6. -0.297 7. 0.82 8. 0.74 9. n = 8 10. 10

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